INSTITUTE OF ECONOMIC STUDIES

Faculty of social sciences of Charles University

Fixed Income Securities

Lecturer's Notes No. 3

Course: Financial Market Instruments II

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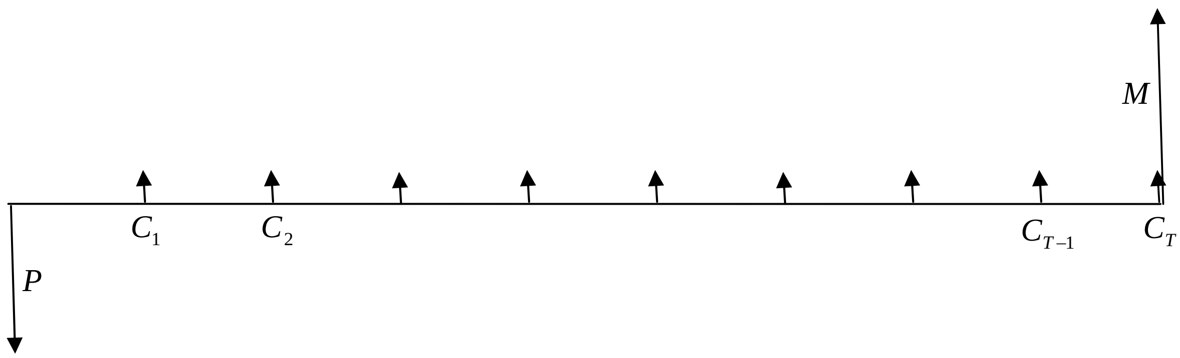
I. BOND CONTRACT

bonds are capital market instruments and as such have maturities in excess of one year

straight bond (plain vanilla bond, bullet bond) pays a regular (usually semi-annual) fixed coupon over a fixed period to maturity (redemption) with the return of principal (par value, nominal value) on the maturity date



bonds may have more intricate flow patterns and can be classified according to many criteria

1.1 Classification of bonds

<u>frequency of coupon payments</u>

- conventional (straight, plain vanilla, bullet) bond pays regular coupons quarterly, semi-annually or annually
- *zero-coupon bond* do not pay coupons at all, they are sold at a deep discount to par value and all reward from holding the bond comes in the form of capital gain
- *income bond* makes coupon payment only if the income generated by the issuing firm is sufficient

<u>size of coupon payments</u>

- conventional bond pays a fixed coupon (determined as a percentage of its par value) over the whole life of the bond
- *variable coupon bond* links its coupon to some economic variable such as the retail price index (index-linked bond), current market interest rate (floating rate note)
- *collateralised bond* derives its coupons from a given package of underlying assets (ABS, asset-backed securities)

redemption date

- conventional bond has one redemption date at maturity (five-year bond, ten-year bond, etc.)
- *double-dated bond* has a range of possible redemption dates
- *callable bond* has an option feature that gives the issue the right to determine the actual date of redemption
- *puttable bond* has an option feature that gives the holder the right to determine the actual date of redemption
- *consol* (*perpetual bond, perpetuity*) have no redemption date at all and interest on it will be paid indefinitely
- *convertible bond* has an option feature that gives the holder the right to convert the bond at maturity into other types of bonds or into equity

issuer of the bond

- *government bond* is issued by a sovereign government in order to finance and manage public debt (Treasuries in USA, gilts in UK)
- *municipal bond* is issue by a local authority
- *corporate bond* is issued by a private company (senior vs. junior debt, secured vs. unsecured debt, fixed vs. floating charge debt)

currency of denomination

- *domestic bond* is issued by a resident issuer in the domestic currency
- *foreign bond* is issued by a non-resident issuer in the domestic currency

    Samurai (Japan), Yankee (USA), Bulldog (UK), Matador (Spain), Kiwi (New Zealand), Alpine (Switzerland)

- *eurobond* is issued and traded in a non-resident currency (other than sterling in the UK, other than euro in the Eurozone countries, etc.)

    many innovations:

    dual currency bond: coupon payment are in one currency and the redemption proceeds are in another

    currency change bonds: coupons are first paid in one currency and then in another

    deferred coupon bond: there is a delay in the payment of the first coupon

    multiple-coupon bond: coupons change over the life of the bond in a predetermined manner

    missing coupon bond: coupon payment is missed whenever a dividend payment on the issuer's shares is missed *refractable bond*: attached with call and put options

bull or bear bond: principal on redemption depends on how a stock index has performed

<u>default or credit risk</u>

it is usually assessed in the form of credit rating (Standard & Poor's, Moody's, Fitch)

the lower the credit rating, the greater the risk of default and the higher the risk premium

grades of the bond: investment grades, non-investment (speculative) grades (*junk bonds*, *high-yielding bonds*)

1.2 Fair pricing of straight bonds

fair price of a bond is given by the discounted present value of the cash flow stream using the market-determined discount rate for a bond of this maturity and risk class

P ... fair price of the bond

M ...nominal value (par value, principal) of the bond

c ... coupon rate (in %)

C… annual fixed coupon payment (= $c \times M$)

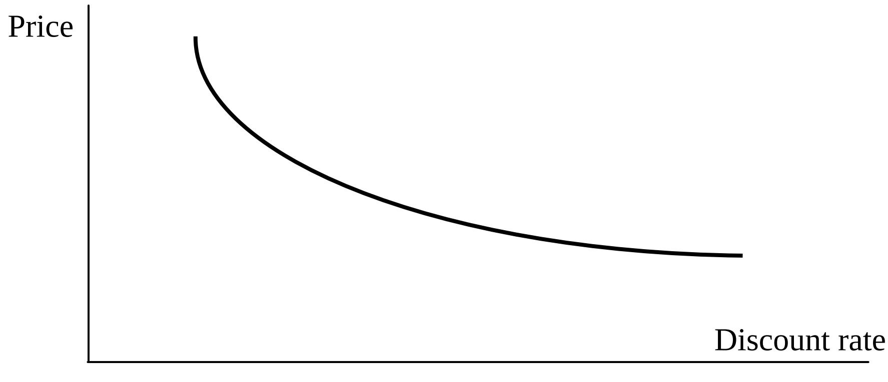
T …number of years to maturity

r … market-determined discount rate

a) annual discounting of annual coupon payments:

$$P = \sum_{t=1}^{T} \frac{cM}{(1+r)^t} + \frac{M}{(1+r)^T}$$
$$= M \times \left[\frac{c}{r} \times \left(1 - \frac{1}{(1+r)^T} \right) + \frac{1}{(1+r)^T} \right]$$

Inverse and convex relationship between the bond price and the discount rate



<u>fair price of a par bond</u>

when the coupon rate is equal to the discount rate the price of the bond will be equal to its par value

$$c = r \quad \Rightarrow \quad P = M$$

bonds with a coupon rate greater than the discount rate are priced below its par value and are said to be selling at a discount

$$c > r \quad \Rightarrow \quad P < M$$

bonds with a coupon rate smaller than the discount rate are priced above its par value and are said to be selling at a premium

$$c < r \quad \Rightarrow \quad P > M$$

<u>fair price of a perpetuity</u>

$$T = \infty \quad \Rightarrow \quad P = M \times \frac{c}{r} = \frac{C}{r}$$

fair price of a bond when valuation date differs from coupon payment date

the bond's cash flow is discounted back to the date of the next coupon payment and then the total sum is discounted to the present day

$$P = \frac{1}{(1+r)^{n/360}} \times \left(cM + \frac{cM}{1+r} + ... + \frac{cM}{(1+r)^T} + \frac{M}{(1+r)^T} \right)$$
$$= \frac{M}{(1+r)^{n/360}} \times \left[c + \frac{c}{r} \times \left(1 - \frac{1}{(1+r)^T} \right) + \frac{1}{(1+r)^T} \right]$$

n … number of days to the next coupon date

b) semi-annual discounting of semi-annual coupon payments:

$$P = \frac{C/2}{\left(1+\frac{r}{2}\right)} + \frac{C/2}{\left(1+\frac{r}{2}\right)^2} + \ldots + \frac{C/2 + M}{\left(1+\frac{r}{2}\right)^{2T}}$$

c) semi-annual discounting of annual coupon payments:

$$P = \frac{C}{\left(1+\frac{r}{2}\right)^2} + \frac{C}{\left(1+\frac{r}{2}\right)^4} + \ldots + \frac{C + M}{\left(1+\frac{r}{2}\right)^{2T}}$$

d) annual discounting of semi-annual coupon payments:

$$P = \frac{C/2}{(1+r)^{1/2}} + \frac{C/2}{(1+r)} + \ldots + \frac{C/2+M}{(1+r)^T}$$

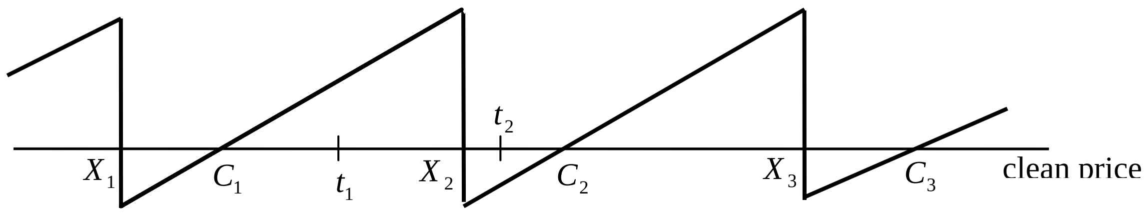
1.3 Clean and dirty price

full (dirty, gross) price of a bond is the price that is actually paid for the bond

it should be equal to the present value of all future cash flows obtained by the holder of the bond

clean price is the quoted price of a bond that disregards the interest accrued from the last coupon payment date

full price = clean price + accrued interest



C_i … coupon payment date

X_i … ex-dividend date

t_i …transaction date (the bond has been sold to a new owner)

between the coupon payment date and the next ex-dividend date the bond is traded *cum dividend*

cum dividend means that the issuer will mail the entire coupon to a new bondholder (he will be registered as the owner of the bond)

the buyer of the bond must compensate the seller for giving up the entire next coupon even though the seller held the bond for part of the period since the last coupon payment

full price = clean price $+ cM \times \frac{t_1 - C_1}{360}$ (full price > clean price)

between the ex-dividend date and the next coupon payment date the bond is traded *ex dividend* ex dividend means that the issuer will mail all the coupon to an original bondholder (he may still be registered as the owner of the bond)

the seller of the bond must compensate the buyer for giving up all of the next coupon even though the buyer will be holding the bond for part of the period since the last coupon payment

$$\text{full price} = \text{clean price} - cM \times \frac{C_2 - t_2}{360} \quad (\text{full price} < \text{clean price})$$

♦

A Treasury bond pays 9 % coupon annually. On 12 June the bond has 63 days to the next coupon payment and there are 297 days since the last coupon payment (assuming 360 days in a year). Since the next coupon payment the bond will have 5 years to maturity. The current market yield for the bond is 8 %.

full price =

$$\frac{100}{(1.08)^{63/360}} \times \left[0.09 + \frac{0.09}{0.08} \times \left(1 - \frac{1}{1.08^5}\right) + \frac{1}{(1.08)^5}\right] = \frac{100}{1.0136} \times 1.1299 = 111.474$$

accrued interest = $0.09 \times 100 \times \frac{297}{360} = 7.425$

clean price = $111.474 - 7.425 = 104.05$

What are the calculations if the current market yield for the bond is 9 % (equal to the coupon rate)?

full price =

$$\frac{100}{(1.09)^{63/360}} \times \left[0.09 + \frac{0.09}{0.09} \times \left(1 - \frac{1}{1.09^5}\right) + \frac{1}{(1.09)^5}\right] = \frac{100}{1.0152} \times 1.0900 = 107.368$$

accrued interest = $0.09 \times 100 \times \frac{297}{360} = 7.425$

clean price = $107.368 - 7.425 = 99.94$

The clean price of the bond is not exactly equal to the notional amount despite the fact that the discount rate is equal to the coupon rate. This is because accrued interest is calculated on a simple interest basis and price on a compound interest basis.

♦

1.4 Yield measures on bonds

because of complicated patterns of cash flows the bonds are compared in terms of yields (rate of return earned from investing in holding the bond)

a) ***yield to maturity*** (YTM, redemption yield, internal rate of return)

YTM is the most frequently used measure of the return from holding the bond

YTM is a discount rate at which the present value of the discounted cash flows is equal to the current dirty price of the bond

YTM is the solution of the equation (has to be found through numerical iteration using a computer calculator)

$$P = \sum_{t=1}^{T} \frac{cM}{(1+\text{YTM})^t} + \frac{M}{(1+\text{YTM})^T}$$

assumptions: annual discounting of annual coupon payments, the present date is the coupon payment date

shortcomings of YTM:

i) YTM is a measure of a bond yield that is based on the implicit assumption that each coupon can be reinvested at a constant YTM rate (this rate of interest is constant for all future time periods)

YTM of a bond completely ignores the reinvestment risk (as opposed to the YTM of a time deposit with compounding interest)

rearranged fair price equation:

$$\text{YTM} = \sqrt[T]{\frac{C(1+\text{YTM})^{T-1} + C(1+\text{Y}TM)^{T-2} + \ldots + C(1+\text{YTM}) + C + M}{P}} - 1$$

nominator: terminal value of cash flows from holding a bond under assumption that all coupons are reinvested at an YTM rate for the remaining life of the bond

denominator: bond price equal to initial value of the investment

ii) YTM is based on the implicit assumption that bond is held to maturity

b) ***holding period yield***

HPY is the average yield realized during the holding (investment) period, taking into account the purchasing and selling prices of the bond as well as changes in the rollover rates at which coupon payments can be reinvested

HPY is the solution of the equation

$$P_B(1+\mathrm{HPY})^T = C(1+r_1)^{T-1} + C(1+r_2)^{T-2} + \ldots + C + P_S$$

P_B ... buying price of the bond

P_S ... selling price of the bond

r_t ... rollover (reinvestment) rate

disadvantage: HPY used as an ex ante measure of the bond's attractiveness requires making assumptions about rollover rates that are uncertain

♦

A bond is purchased on a coupon payment date for 96.50 and sold exactly two years later for 101.30. The annual coupon is 8.7 and is paid semi-annually. The rollover rates for the first three coupons are 10.00, 10.25 and 10.40 % respectively.

The holding period yield can be found by solving the equation

$$96.50\times\left(1+\frac{\mathrm{HPY}}{2}\right)^4 = \frac{8.7}{2}\times\left[\left(1+\frac{0.10}{2}\right)^3+\left(1+\frac{0.102}{2}\right)^2+\left(1+\frac{0.104}{2}\right)\right]+101.30$$

$$= 14.42 + 101.30 = 115.72$$

$$\mathrm{HPY} = \left[\left(\frac{115.72}{96.50}\right)^{1/4} - 1\right]\times 2 = 0.093 = 9.3\ \%$$

♦

c) ***current yield*** (flat yield, running yield, interest yield, income yield)

CY is the simplest measure of the rate of return on a bond

$$\mathrm{CY} = \frac{\text{annual coupon}}{\text{price of the bond}}$$

more appropriate is the dirty price of the bond price which is the purchasing price but the clean price does not result in the saw-tooth pattern of the CY the reciprocal value of the CY indicates a number of coupon payments that will fully repay the purchasing price of the bond

disadvantage: CY does not take into account capital gains or losses resulting from the differences between the current price of the bond and its maturity value (more suitable for long-dated bonds with small capital gain/loss element)

d) ***simple yield to maturity***

assumption is made that the capital gain/loss occurs evenly over the remaining life or the bond

$$\text{SYM} = \text{CY} + \frac{100 \text{ - bond price}}{\text{years to maturity} \times \text{bond price}}$$

♦

A bond pays an annual coupon of £ 8.75 and its current clean price is £ 95.30. The bond has 4 years to maturity.

$$\text{CY} = \frac{8.75}{95.30} = 9.18\ \%$$

$$\text{SYM} = \text{CY} + \frac{100 - 95.30}{4 \times 95.30} = 9.18 + 1.23 = 10.41\ \%$$

♦

e) ***money market yield***

yields for bonds that have only one remaining coupon to be paid at maturity are sometimes quoted on a basis comparable to money market instruments (simple interest, actual number of days)

capital market conventions (compound interest)

$$P = \frac{C+M}{(1+r)^{n/365}} \quad \Rightarrow \quad r = \left(\frac{C+M}{P}\right)^{365/n} - 1$$

money market conventions (simple interest)

$$P = \frac{C+M}{1+\frac{n}{365}\times r} \quad \Rightarrow \quad r = \left(\frac{C+M}{P} - 1\right) \times \frac{365}{n}$$

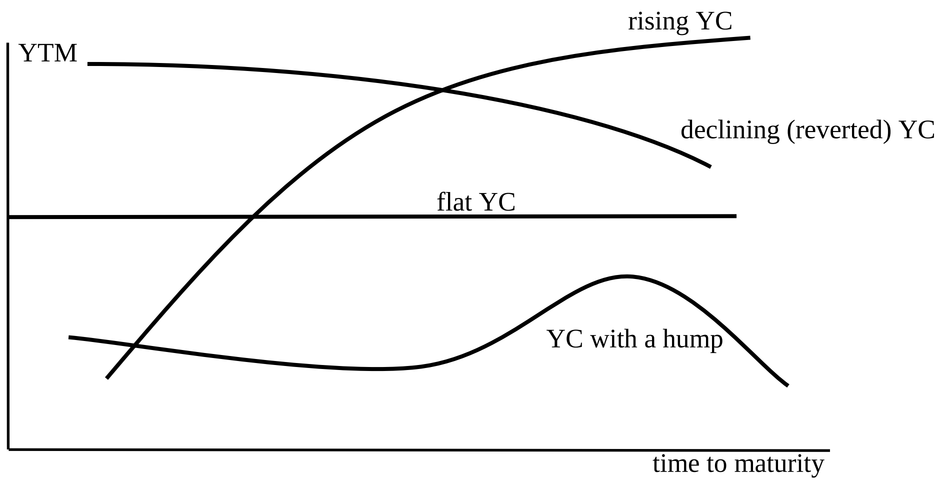
II. ANALYSIS OF THE YIELD CURVE

yield curve (also called ***term structure of interest rates***) is the relationship between a particular yield measure and a bond's maturity

only bonds from a homogeneous group should be included (from the same risk class or with the same degree of liquidity)

YTM is the most frequent measure used in constructing yield curves $\Rightarrow$ yield curve is a plot of the bonds' yield to maturity against their term to maturity

Shapes of yield curve

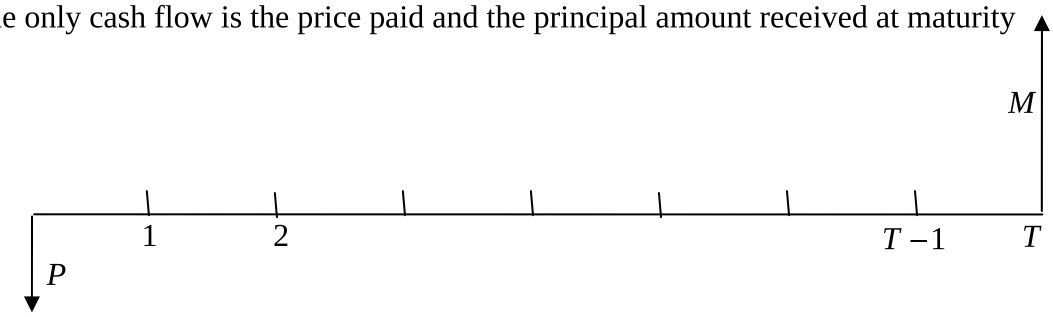


empirical yield curve is a plot of the YTM of existing bonds against their term to maturity problems:

- YTM measure applied on coupon-bearing bonds suffers from reinvestment risk
- YTM curve does not distinguish between the different payment patterns of low-coupon and high-coupon bonds with the same maturity (they may have different YTMs because they may differ in terms of their attractiveness for investors)

2.1 Zero yield curve

zero-coupon bonds (***pure discount bonds***) are bonds that make no interest payment and the only cash flow is the price paid and the principal amount received at maturity



zero rates (***spot rates***) are YTMs earned on zero-coupon bonds

zero rates are the annual interest rates at which the discounted value of the principal amount equals the bond's price

zero rate of *t*-year zero-coupon bond

$$P = \frac{M}{(1+z_t)^t} \quad \Rightarrow \quad z_t = \left(\frac{M}{P}\right)^{1/t} - 1$$

a clear advantage of zero-coupon yields is that they avoid completely the question of reinvestment risk (because of this certainty investors may accept a slightly lower overall yield)

zero yield curve (***spot yield curve***) is a plot of zero rates of zero-coupon bonds against their term to maturity

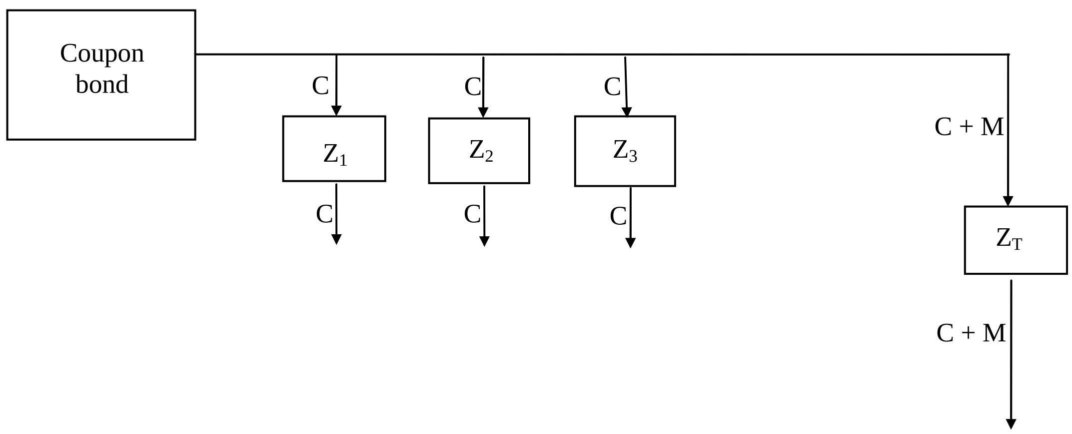
there may be no complete set of zero-coupon bonds available for the construction the zero yield curve, but there must be theoretical zero rates which are consistent with the interest rates observed in the market

bootstrapping is the analytical method of building up a series of consistent zero yields from coupon-bearing yields

practical counterparts of bootstrapping:

- *bond stripping* (*unbundling*) is a technique of financial engineering that breaks down the coupon-bearing bond into coupons and principal components as separate securities that are traded separately
- *synthetic zero-coupon bond* is a technique of financial engineering that combines a series of coupon-bearing bonds in such a way that all the cash flows are zero except for the first and the last

Bond stripping



derivation of the zero yield curve

$r_1, r_2, ..., r_T$ are yields to maturity of observed coupon-bearing bonds (the bonds are assumed to be par bonds but any series of bonds can be used for bootstrapping)

$z_1, z_2, ..., z_T$ are zero yields of theoretical zero-coupon bonds consistent with the given set of coupon-bearing yields

• zero rate for one-year maturity

one-year bond is already by assumption a zero-coupon bond, therefore

$$z_1 = r_1$$

• zero rate for two-year maturity – method of bond stripping

present value of two-year coupon-bearing par bond (with principal 1):

$$V_1 = \frac{r_2}{1+r_2} + \frac{r_2+1}{(1+r_2)^2} = 1$$

sum of present values of zero-coupon securities stripped form the coupon-bearing bond:

$$V_2 = \frac{r_2}{1+z_1} + \frac{r_2+1}{(1+z_2)^2}$$

no-arbitrage condition implies that both values must be equal ⇒ one can then solve for unknown variable z_2

$$1 = V_1 = V_2 = \frac{r_2}{1+z_1} + \frac{r_2+1}{(1+z_2)^2}$$

• zero rate for two-year maturity – method of synthetic zero-coupon bond

purchasing one two-year par coupon-bearing bond generates the cash flow

$t = 0: \quad -1$

$t = 1: \quad r_2$

$t = 2: \quad 1 + r_2$

issuing one-year bonds in nominal value equal to the discounted coupon of the two-year bond generates the cash flow:

$t = 0: \quad r_2/(1+z_1)$

$t = 1: \quad -r_2$

$t = 2: \quad 0$

combined cashflow generates the cashflow pattern of a two-year zero-coupon bond

$$t = 0: \quad -1 + r_2/(1 + z_1)$$
$$t = 1: \quad 0$$
$$t = 2: \quad 1 + r_2$$

rate of return of the synthetic two-year zero-coupon bond is given by the equation

$$1 + z_2 = \sqrt{\frac{r_2 + 1}{1 - \dfrac{r_2}{1 + z_1}}}$$

it can be demonstrated that the obtained expression is equivalent to the previous expression

$$1 = \frac{r_2}{1 + z_1} + \frac{r_2 + 1}{(1 + z_2)^2}$$

• zero rate for *T*-year maturity

in a similar fashion the bootstrapping process works forward iteratively for all other maturities

with already determined zero rates $z_1, z_2, \ldots, z_{T-1}$ the zero rate z_T can be determined from the equation

$$1 = \sum_{t=1}^{T-1} \frac{r_T}{(1 + z_t)^t} + \frac{r_T + 1}{(1 + z_T)^T}$$

practical obstacle of bootstrapping: it is not generally possible to find a series of existing bonds with the convenient maturities ⇒ one must interpolate between existing yields in order to establish rates for all maturities (assumptions must be made about fitting a curve along existing points)

♦

Suppose that a series of bonds are currently priced as follows:

Bond	**Price (€)**	**Coupon (%)**	**Maturity (years)**
A	90.91	0	1
B	97.41	9	2
C	85.26	5	3
D	104.65	13	4

1. Determination of one-year zero rate:

$$z_1 = \frac{100}{90.91} - 1 = 10\ \%$$

2. Determination of two-year zero rate:

a) bond stripping

$$97.41 = \frac{9}{(1+0.10)^1} + \frac{109}{(1+z_2)^2}$$

$$\Rightarrow \quad z_2 = \left(\frac{109}{89.23}\right)^{1/2} - 1 = 10.52\ \%$$

b) synthetic zero-coupon bond

Bond	**2-year bond**	**1-year bond**	**Total CF**
Quantity	1	0.09	x
$t = 0$	-97.41	8.18	-89.23
$t = 1$	9	-9	0
$t = 2$	109	0	109

$$z_2 = \left(\frac{109}{89.23}\right)^{1/2} - 1 = 10.52\ \%$$

3. Determination of three-year zero rate:

a) bond stripping

$$85.26 = \frac{5}{(1+0.10)^1} + \frac{5}{(1+0.1052)^2} + \frac{105}{(1+z_3)^3}$$

$$\Rightarrow \quad z_3 = \left(\frac{105}{76.62}\right)^{1/3} - 1 = 11.08\ \%$$

b) synthetic zero-coupon bond

Bond	**3-year bond**	**2-year bond**	**1-year bond**	**Total CF**
Quantity	1	0.0459	0.0459	x
$t = 0$	-85.26	4.47	4.17	-76.62
$t = 1$	5	-0.41	-4.59	0
$t = 2$	5	-5	0	0
$t = 3$	105	0	0	105

$$z_3 = \left(\frac{105}{76.62}\right)^{1/3} - 1 = 11.08\ \%$$

4. Determination of four-year zero rate:

a) bond stripping

$$104.65 = \frac{13}{(1+0.10)^1} + \frac{13}{(1+0.1052)^2} + \frac{13}{(1+0.1108)^3} + \frac{113}{(1+z_4)^4}$$

$$\Rightarrow \quad z_4 = \left(\frac{113}{72.70}\right)^{1/4} - 1 = 11.66\ \%$$

b) synthetic zero-coupon bond

Bond	**4-year bond**	**3-year bond**	**2-year bond**	**1-year bond**	**Total CF**
Quantity	1	0.1238	0.1136	0.1136	x
$t = 0$	-104.65	10.56	11.07	10.33	-72.69
$t = 1$	13	-0.62	-1.02	-11.36	0
$t = 2$	13	-0.62	-12.38	0	0
$t = 3$	13	-13	0	0	0
$t = 4$	113	0	0	0	113

$$z_4 = \left(\frac{113}{72.69}\right)^{1/4} - 1 = 11.66\ \%$$

♦

2.2 Implied forward yield curves

forward-forward rate is the interest rate on a cash borrowing or lending that starts on one future date and ends on another future date

$_t f_{t+p}$ … interest rate for the period that starts in *t* years' time and ends in (*t*+*p*) years' time (interest rate for *p* years' period beginning in *t* years' time)

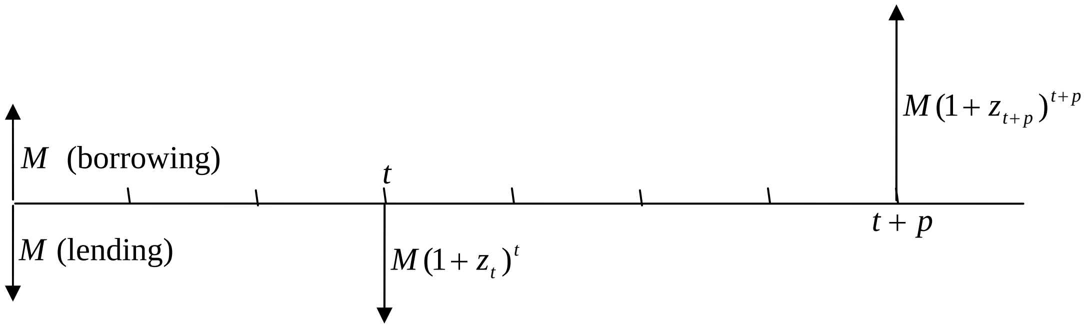
yield to maturity of a *p*-year zero-coupon bond that will be issued in *t*-years' time

<u>building up synthetic forward-forward rates from zero rates</u>

borrowing *M* € for *t* years (cash inflow of *M* at $t = 0$) at z_t generates a cash outflow

$M \times (1+z_t)^t$ at time *t* (repayment of borrowed funds) lending M € for $t + p$ years (cash outflow of M at $t = 0$) at z_{t+p} generates a cash inflow

$M \times (1+z_t)^t$ at time $t + p$ (lent funds are returned back)



net position of the two cash transactions is lending at the year t for $t + p$ years

implied future lending rate can be found from the equation

$$(1+{}_t f_{t+p})^p = \frac{(1+z_{t+p})^{t+p}}{(1+z_t)^t}$$

$$_t f_{t+p} = \left(\frac{(1+z_{t+p})^{t+p}}{(1+z_t)^t} \right)^{1/p} - 1$$

implied forward rates are the interest rates that are consistent with a given zero-coupon yield curve (they are calculated on the basis of this curve)

inconsistent forward rates indicate the existence of an arbitrage opportunity

♦

Suppose that the following zero rates were extracted from a given structure of market bond yields (the same as in previous example):

Maturity	**1**	**2**	**3**	**4**
Zero rate	10.00	10.52	11.08	11.66

1-year rate starting in 1 year's time = $\frac{(1+0.1052)^2}{1+0.10} - 1 = 11.04\,\%$

1-year rate starting in 2 years' time = $\frac{(1+0.1108)^3}{(1+0.1052)^2} - 1 = 12.21\,\%$

2-year rate starting in 2 years' time = $\left(\frac{(1+0.1166)^4}{(1+0.1052)^2} \right)^{1/2} - 1 = 12.81\,\%$

♦

Suppose that the 1-year rate starting in 1 year's time is 11.5 % rather then 11.04 %. Therefore this interest rate is not consistent with given structure of zero yields and the following arbitrage opportunity exists:

Take a 2-year loan, invest the funds into 1-year deposit, after one year reinvest the proceeds into another 1-year deposit, repay the loan.

final cash inflow = $100 \times 1.10 \times 1.115 = 122.65$

final cash outflow = $100 \times 1.1052^2 = 122.15$

arbitrage profit = $122.65 - 122.15 = .50$

♦

forward yield curve is a plot of implied forward rates against term to maturity for a given future starting date

forward yield curve in 1-year time: $_1f_2,\ _1f_3, \ldots,\ _1f_T$ $(T-1$ values)

forward yield curve in 2-years' time: $_2f_3,\ _2f_4, \ldots,\ _2f_T$ $(T-2$ values)

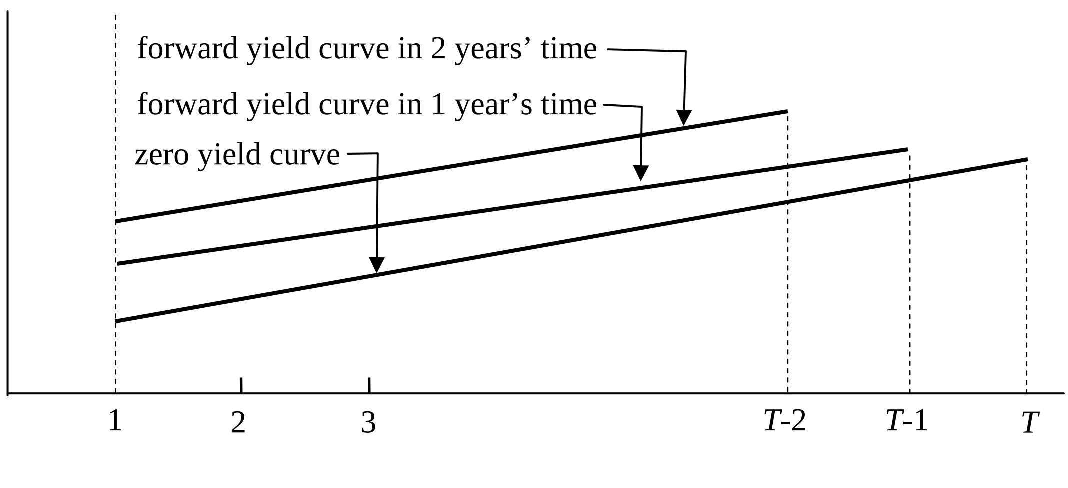
. . . .

forward yield curve in (T-1)-years' time: $_{T-1}f_T$ (1 value)

current yield curve can be defined as the forward yield curve starting now

$$z_1 = {}_0f_1,\ z_2 = {}_0f_2, \ldots,\ z_T = {}_0f_T$$

Family of implied forward yield curves



building up zero rates from forward-forward rates

zero rates can be determined as the geometric mean of the chain of appropriate forward-forward rates (using convention $z_1 = {}_0f_1$)

$$\begin{aligned}(1+z_T)^T &= (1+z_{T-1})^{T-1} \times (1+{}_{T-1}f_T) \\ &= (1+z_{T-2})^{T-2} \times (1+{}_{T-2}f_{T-1}) \times (1+{}_{T-1}f_T) \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &= (1+{}_0f_1) \times (1+{}_1f_2) \times \ldots\ldots \times (1+{}_{T-2}f_{T-1}) \times (1+{}_{T-1}f_T)\end{aligned}$$

♦

The 1-year interest rate is now 10 %. The market expects 1-year interest rate to rise to 11.05 % after one year and to 12.18 % after a further year. What are the current 2-year and 3-year zero-coupon yields consistent with these expectations?

$$z_2 = [(1+{}_0f_1) \times (1+{}_1f_2)]^{1/2} - 1 = \sqrt{1.1 \times 1.1105} - 1 = 10.52\ \%$$

$$z_3 = [(1+{}_0f_1) \times (1+{}_1f_2) \times (1+{}_2f_3)]^{1/3} - 1 = \sqrt{1.1 \times 1.1105 \times 1.1218} - 1 = 11.07\ \%$$

♦

expected changes in interest rates

if the zero rate curve is upward sloping ($z_{t-1} < z_t$) then the market expects the future interest rates to rise

- proof for all future 1-year rates

$$(1+z_{t-1})^t < (1+z_t)^t = (1+z_{t-1})^{t-1} \times (1+{}_{t-1}f_t) \quad \Rightarrow \quad z_{t-1} < {}_{t-1}f_t$$

$$z_1 < z_{t-1},\ z_{t-1} < {}_{t-1}f_t \quad \Rightarrow \quad z_1 < {}_{t-1}f_t \quad \text{for all } t$$

- proof for all next year's interest rates

$$(1+z_t)^t = (1+z_1) \times (1+{}_1f_t)^{t-1} = (1+z_{t-1})^{t-1} \times (1+{}_{t-1}f_t) > (1+z_{t-1})^{t-1} \times (1+z_1)$$

$$\Rightarrow (1+{}_1f_t)^{t-1} > (1+z_{t-1})^{t-1} \quad \Rightarrow \quad z_{t-1} < {}_1f_t \quad \text{for all } t$$

similarly if the zero rate curve is downward sloping ($z_{t-1} > z_t$) then the market expects future interest rates to fall

2.3 Par yield curve

par yield is the yield to maturity of a bond that is priced at or near par (price of the bond is equal to its nominal value)

par yield is equal to the coupon rate for such bond (since YTM for bonds priced at par is equal to the coupon rate)

par yields consistent with a given structure of zero rates can be found by solving the equation

$$M = \sum_{t=1}^{T} \frac{c_T M}{(1+z_t)^t} + \frac{M}{(1+z_T)^T}$$

$$c_T = \frac{1-d_T}{\sum_{t=1}^{T} d_t} \quad \text{where} \quad d_t = \frac{1}{(1+z_t)^t}$$

d_t is the discount factor for the *t*-year zero-coupon bond with the YTM of z_t

♦

Determine par yields consistent with the following structure of zero-coupon yields?

Maturity	**1**	**2**	**3**	**4**
Zero rate	10.00	10.52	11.08	11.66
Discount factor	0.909091	0.818688	0.729613	0.643229

$$c_1 = \frac{1-d_1}{d_1} = 10.00\,\%, \; c_2 = \frac{1-d_2}{d_1+d_2} = 10.49\,\%$$

$$c_3 = \frac{1-d_3}{d_1+d_2+d_3} = 11.00\,\%, \; c_4 = \frac{1-d_4}{d_1+d_2+d_3+d_4} = 11.51\,\%$$

♦

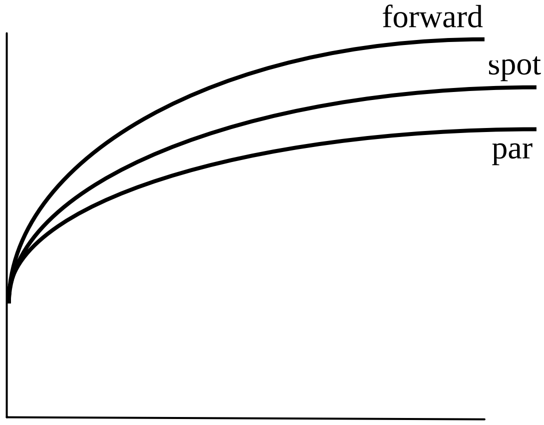
par yield curve is a plot of the yield to maturity against term to maturity for bonds priced at par

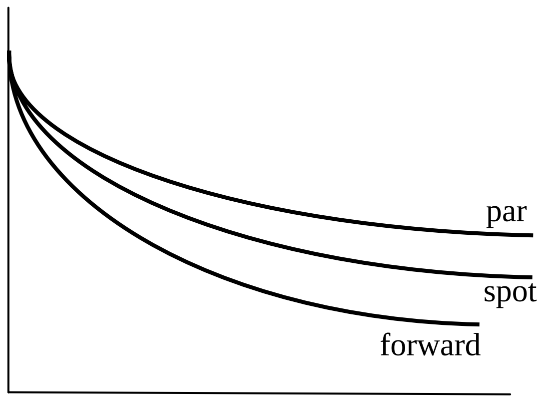
if the zero rate curve is upward sloping ($z_{t-1} < z_t$) then the par yield curve lies below the zero rate curve

if the zero rate curve is downward sloping ($z_{t-1} > z_t$) then the par yield curve lies above the zero rate curve proof for the upward sloping zero curve

$$c_T = \frac{1-d_T}{\sum_{t=1}^{T} d_t} = \frac{1-\frac{1}{(1+z_T)^T}}{\sum_{t=1}^{T}\frac{1}{(1+z_t)^t}} < \frac{1-\frac{1}{(1+z_T)^T}}{\sum_{t=1}^{T}\frac{1}{(1+z_T)^t}} = \frac{1-\frac{1}{(1+z_T)^T}}{\frac{1}{z_T}\times\left[1-\frac{1}{(1+z_T)^T}\right]} = z_T$$

Position of par, spot and forward yield curves



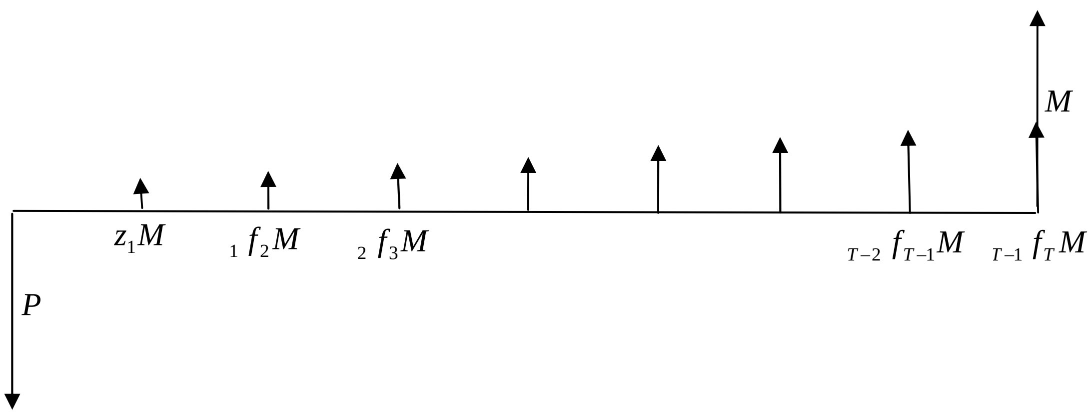


2.4 Pricing of floating rate notes

FRN (floaters) are types of bonds (capital market securities) whose coupon payments are variable and related to a given short-term market rate (i.e. 3-month LIBOR, 6-month LIBOR)

coupons of the FRN are fixed for a given period and are reset on the ***coupon reset dates***

fair price of FRN = expected cash flow (z_1M, ${}_1f_2M$, ${}_2f_3M$,..., ${}_{T-1}f_TM + M$) discounted at zero rates of interest of corresponding maturities $\Rightarrow$ FRN are generally issued at par



fair price of FRN on the coupon reset day

relationship between zero and forward rates: $(1+z_t)^t(1+{}_t f_{t+1}) = (1+z_{z+1})^{t+1}$

$$
\begin{aligned}
P_{\text{FRN}} &= \frac{z_1 M}{(1+z_1)} + \frac{{}_1 f_2 M}{(1+z_2)^2} + \ldots + \frac{{}_{T-1} f_T M}{(1+z_T)^T} + \frac{M}{(1+z_T)^T} \\
&= M\left[\frac{z_1}{(1+z_1)} + \sum_{t=1}^{T-1} \frac{1}{(1+z_{t+1})^{t+1}}\left[\frac{(1+z_{t+1})^{t+1}}{(1+z_t)^t} - 1\right]\right] + \frac{M}{(1+z_T)^T} \\
&= M\left[\frac{z_1}{(1+z_1)} + \sum_{t=1}^{T-1}\left[\frac{1}{(1+z_t)^t} - \frac{1}{(1+z_{t+1})^{t+1}}\right]\right] + \frac{M}{(1+z_T)^T} \\
&= M\left[\frac{z_1}{(1+z_1)} + \frac{1}{(1+z_1)} - \frac{1}{(1+z_T)^T}\right] + \frac{M}{(1+z_T)^T} \\
&= M - \frac{M}{(1+z_T)^T} + \frac{M}{(1+z_T)^T} = M
\end{aligned}
$$

the floater is sold at par (for its nominal value)

2.5 Inflation-indexed bond

inflation-indexed (inflation-linked) bond is one whose cash flows are linked to movements in a specific price index (i.e. CPI) with the aim of providing protection for real value of investment

bonds are issued with a certain real rate of return (i.e. 2 – 2.5 %) while a nominal rate of will depend on the realized future path of the price index

coupon payment = coupon rate × principal × $\frac{I_t}{I_0}$

repayment of principal = principal × $\frac{I_T}{I_0}$

I_0, I_t, I_T are values of the price index i) when the bond is issued; ii) when the bond pays regular coupons iii) when the bond matures

inflation-linked bonds offer only imperfect protection against erosion of purchasing power

- the weighted composition of a general price index does not coincide with the collection of goods consumed by investors who want to be protected against inflation
- statistical reporting and procession lags (up to 2 months) mean that there is always a delay between the relevant time period for which an index is computed and the date on which that number is published

- calculation of accrued interest requires to fix the nominal value of the price index at the beginning of the six-month coupon period, therefore coupon or principal payments are based on at least a six-month old price index

nominal rate of return (YTM) on an inflation-linked bond

depends on an uncertain forecast of the inflation rate π

$$
\begin{aligned}
P &= \frac{C(I_1/I_0)}{1+r} + \frac{C(I_2/I_0)}{(1+r)^2} + \ldots + \frac{(C+M)(I_T/I_0)}{(1+r)^T} \\
&= \frac{C(1+\pi)}{(1+r)} + \frac{C(1+\pi)^2}{(1+r)^2} + \ldots + \frac{(C+M)(1+\pi)^T}{(1+r)^T}
\end{aligned}
$$

Fisher equation

$$(1+r) = (1+\rho) \times (1+\pi)$$

r … nominal yield, ρ … real yield, π … inflation rate

real rate of return (YTM) on an inflation-linked bond

$$
\begin{aligned}
P &= \frac{C(1+\pi)}{(1+r)} + \frac{C(1+\pi)^2}{(1+r)^2} + \ldots + \frac{(C+M)(1+\pi)^T}{(1+r)^T} \\
&= \frac{C(1+\pi)}{(1+\rho)(1+\pi)} + \frac{C(1+\pi)^2}{(1+\rho)^2(1+\pi)^2} + \ldots + \frac{(C+M)(1+\pi)^T}{(1+\rho)^T(1+\pi)^T} \\
&= \frac{C}{(1+\rho)} + \frac{C}{(1+\rho)^2} + \ldots + \frac{(C+M)}{(1+\rho)^T}
\end{aligned}
$$

break-even inflation is the inflation that makes the money yield on an inflation-linked bond equal to the yield on a conventional bond of the same maturity

break-even inflation can be determined using the Fisher equation

♦

Suppose that the yield to maturity on a conventional Treasury is 8.3 % and that the inflation-linked Treasury of the same maturity is 2.5 %. Coupons are paid semi-annually. Determine the break-even inflation.

$$1+\pi = \frac{\left(1+\frac{r}{2}\right)^2}{\left(1+\frac{\rho}{2}\right)^2} = \frac{\left(1+\frac{0.083}{2}\right)^2}{\left(1+\frac{0.025}{2}\right)^2} = 1.0581$$

At an inflation of 5.8 % the inflation-linked bond is equally attractive as the conventional bond.

♦

if the expected rate of inflation is higher than the break-even inflation then investors will prefer the index-linked bonds to conventional ones

excess demand for these bonds will eventually press down the yield of inflation protected bond to the level of conventional ones (and vice versa)

in efficient markets the break-even inflation thus reveals the inflation that is expected by market participants

2.6 Theories of the yield curve

main task of these theories is to explain what determines the position and the shape of the yield curve

expectation hypothesis argues that

a) current long-term interest rates are related to expected future short-term interest rates (in accordance with the theory behind the forward yield curve)

$$\begin{aligned}(1+z_T)^T &= (1+z_t)^t \times (1+{}_t f_T)^{T-t} \\ &= (1+{}_0 f_1)\times(1+{}_1 f_2)\times\ldots\ldots\times(1+{}_{T-2} f_{T-1})\times(1+{}_{T-1} f_T)\end{aligned}$$

b) implied forward interest rates are the best indicator of expected future interest rates

$${}_t f_{t+p} = E_t(z_p)$$

${}_t f_{t+p}$ … current p-year forward rate starting in t years' time

$E_t(z_p)$ ... future p-year sport rate expected at the year t

the market believes that at time t the p-year interest rate will have a value of ${}_t f_{t+p}$

current interest rates are adjusted to the expected values of future interest rates

♦

The current one-year rate is 6.5 % and the market is expecting the one-year rate in a year's time to be 7.5 %. The expectation hypothesis maintains that the current two-year rate will be

$$z_2 = \sqrt{1.065\times1.075} - 1 = 0.07 = 7\ \%$$

If the market reassess the interest rate expectation for the second year from 7.5 % to 8.0 %, the current two-year spot rate changes immediately to a value of

$$z_2 = \sqrt{1.065 \times 1.08} - 1 = 0.0724 = 7.24\ \%$$

♦

components of the nominal interest rate

nominal interest rate = real interest rate + inflation premium + liquidity premium + risk premium

real interest rate is given by the rate of time preferences or by the willingness of the individuals to forgo current consumptions in return for additional consumption in the future periods

real interest rate is the price paid for the willingness to transfer resources over time and equates the supply of funds for those willing to lend with the demand for funds for those wishing to borrow

inflation premium is the price paid to lenders for the erosion of the purchasing power of provided funds that will be caused by an expected inflation

inflation premium allows to buy in the future the same bundle of goods that could be bought when money were provided

Fisher equation:

(1 + nominal interest rate) = (1 + real interest rate) × (1 + inflation rate)

$$(1 + r) = (1 + \rho) \times (1 + \pi)$$
$$r \doteq \rho + \pi$$

liquidity premium is related to the effect of the duration of lending and borrowing

liquidity preference theory:

most lenders prefer to lend for short periods because short-term lending can be more easily converted to cash without the risk of losing capital value; in order to induce lenders to provide long-term funding they have to be offered a liquidity premium on long money

most borrowers prefer to borrow for long periods because short-term borrowing has to be rolled over and there is the risk that this can be done on unfavourable terms; borrowers are willing to pay a liquidity premium on long money preferences of both lenders and borrowers thus combine to give the upward-sloping yield curve (even if the real interest rate and the inflation rate are not expected to change over time)

liquidity premium is likely to increase with the term to maturity but at a decreasing way (the curve initially rise steeply and then flattens out)

segmentation (preferred habitat) theory:

financial markets are segmented by maturity range and there are no spillover effects between each market segment

examples: pension funds are active in long-term bonds, banks are more interested in short-term securities, specific role of government policy in securing public sector borrowing and managing public debt, central bank policies in controlling money supply, etc.)

the shape of the yield curve is determined by supply and demand conditions in each market segment without reference to conditions in other segments

risk premium is related to the risk that the issuer of a bond will default on his obligations

specific risk results from the events that are directly linked to a particular issuer (incompetent decisions made by managers, insufficient sales revenue to finance operating cost, inadequate collateral in the event of insolvency, etc.)

market risk results from the dependence of firm's earnings on market conditions

III. MEASUREMENT OF INTEREST RATE RISK

3.1 Risks associated with investing in bonds

***interest rate risk* (*price risk, market risk*)** is the risk of a capital loss for an investor who may have to sell the bond before the maturity date (as interest rates rise the price of bonds will fall)

this risk is not of concern for an investor who plans to hold a bond to maturity

reinvestment risk is the risk that interim coupons of a bond will have to be reinvested at a lower rate such that the investor will earn a lower yield than the stated yield at the time the bond is purchased

***default risk* (*credit risk*)** is the risk that the issuer of a bond will default on its contractual payments of interest rate and/or principal

the risk of default is gauged by quality ratings assigned by rating companies (or credit research staff of financial firms)

inflation risk is the risk that the return realized from investing in a bond will not be sufficient to offset the loss in purchasing power due to inflation

this risk is not of concern for adjustable or floating rate bonds

call risk is the risk that the issuer may exercise the contractual right to retire all or part of the issue before the maturity date

disadvantages of call provisions from the investor's perspective:

- cash flow pattern of the bond is not known with certainty
- proceeds from retiring the bond will have to be reinvested at a lower interest rate because bonds are called when interest rates have dropped
- capital appreciation potential of the bond may be reduced because of a given call price stipulated in a bond contract

***foreign exchange risk* (*currency risk*)** occurs when the issuer promises to make payments in a foreign currency so the cash flow will depend on the foreign-exchange rate at the time the cash flow is received

***liquidity risk* (*marketability risk*)** involves the ease with which an issue can be sold at or near the prevailing market price

the greater the dealer spread (difference between the bid price at which the issue can be sold and the ask price at which the issue can be purchased) the greater the liquidity risk

3.2 Duration

duration is defined as the weighted average maturity of a bond using the relative discounted cash flows in each period as weights

the measure can be understood as the average time taken to receive the cash flows on a given bond ⇨ the measuring unit of duration is a year

formula for Macaulay duration (D)

$$D = \sum_{t=1}^{T} w_t \times t = \sum_{t=1}^{T} \frac{C/(1+r)^t}{P} \times t + \frac{M/(1+r)^T}{P} \times T$$
$$= \frac{C}{P} \times \sum_{t=1}^{T} \frac{t}{(1+r)^t} + \frac{M}{P} \times \frac{T}{(1+r)^T}$$

the fair price equation ensures that the sum of weights equals one

$$\sum_{t=1}^{T} w_t = \frac{1}{P}\left[\sum_{t=1}^{T} \frac{C}{(1+r)^t} + \frac{M}{(1+r)^T}\right] = \frac{1}{P} \times P = 1$$

duration can be computed using the analytical formula

$$D = \frac{C}{P} \times \frac{(1+r)^{T+1} - (1+r) - rT}{r^2 (1+r)^T} + \frac{M}{P} \times \frac{T}{(1+r)^T}$$

formula for modified duration (D_M)

$$D_m = \frac{\text{Macaulay duration}}{1+r}$$

♦

A bond with a yield to maturity of 8 % and a coupon of 5 % paid annually has five years to maturity. Calculate the duration of the bond.

$$P = \frac{5}{1.08} + \frac{5}{1.08^2} + \frac{5}{1.08^3} + \frac{5}{1.08^4} + \frac{105}{1.08^5} = \frac{5}{0.08} \times \left(1 - \frac{1}{1.08^5}\right) + \frac{100}{1.08^5} = 88.02$$

$$D = \frac{5}{88.02} \times \left(\frac{1}{1.08} + \frac{2}{1.08^2} + \frac{3}{1.08^3} + \frac{4}{1.08^4} + \frac{5}{1.08^5}\right) + \frac{100}{88.02} \times \frac{5}{1.08^5}$$
$$= \frac{5}{88.02} \times \frac{1.08^6 - 1.08 - 0.08 \times 5}{0.08^2 \times 1.08^5} + \frac{100}{88.02} \times \frac{5}{1.08^5} = 4.51$$

The bond has Macaulay duration of 4.51 years. Modified duration is 4.18 years.

♦

properties of duration

a) duration is always less than (or equal to) maturity

$$D \le T$$

explanation: some weight is given to the cash flows in the early years of the bond's life that helps to bring forward the average time at which the cash flows are received

duration equals to maturity for zero-coupon bonds

duration of perpetuity ($T = \infty$)

$$P = \frac{C}{r}, \quad D = \frac{C}{P} \times \frac{1+r}{r^2} = 1 + \frac{1}{r}$$

b) duration decreases as coupon increases (and vice versa)

explanation: as coupon rises then more of the relative weight of the cash flows is transferred to the present date

c) duration decreases as yield to maturity increases

explanation: as yield rises then the present value of all future cash flows falls but the present value of the more distant cash flows falls relatively more than that of the nearer cash flows

d) duration is the measure of interest rate risk (elasticity of the bond price with respect to one plus the yield to maturity)

proof:

$$P = \sum_{t=1}^{T} \frac{C}{(1+r)^t} + \frac{M}{(1+r)^T}$$

$$\frac{dP}{d(1+r)} = -C \sum_{t=1}^{T} \frac{t}{(1+r)^{t+1}} - \frac{M \times T}{(1+r)^{T+1}}$$

$$\frac{dP/P}{d(1+r)/(1+r)} = \frac{1+r}{P} \times \frac{dP}{d(1+r)} = \frac{1+r}{P} \times \left[-C \sum_{t=1}^{T} \frac{t}{(1+r)^{t+1}} - \frac{M \times T}{(1+r)^{T+1}} \right] = -D$$

the lower the duration, the less responsive is the bonds value to the interest rate fluctuations

duration can be regarded as a first-order approximation of a change in the bond's value with respect to the change in the yield (it measure the slope of the present-value profile)

$$\Delta P \doteq -P \times \frac{\Delta r}{1+r} \times D = -P \times \Delta r \times D_m$$

♦

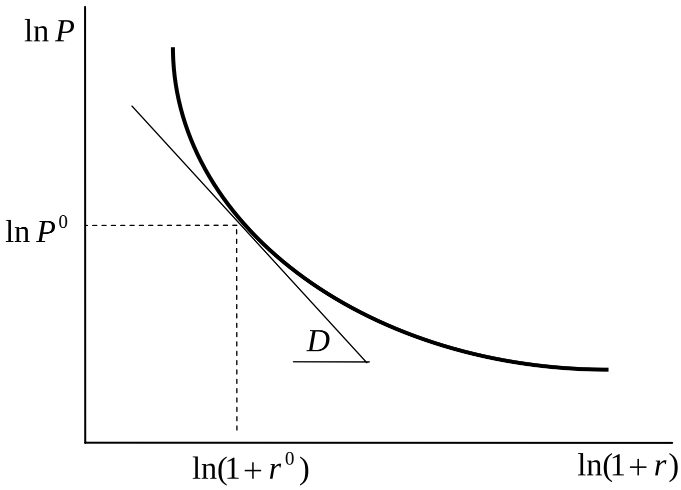
Using the duration measure estimate a new price of the bond from the previous example if yields on five-year bonds are expected to rise from 8 to 10 %?

$$\Delta P \doteq -P \times \frac{\Delta r}{1+r} \times D = -88.02 \times \frac{(0.10-0.08)}{1+0.08} \times 4.51 = -7.35$$

A bond price is expected to fall from 88.02 to (88.02 – 7.35) = 80.67. A direct calculation based on the fair price formula shows that at a yield of 10 % an exact new price would be 81.05. Duration thus overestimates a true fall of the bond's price.

♦

Present-value profile and duration



logarithmic derivation

$$D = -\frac{dP/P}{d(1+r)/(1+r)} = -\frac{d\ln P}{d\ln(1+r)}$$

the greater is the change in the bond's price the greater is the approximation error

e) immunization property of duration

when interest rates rise the investor is exposed to two offsetting effects

accumulated value of reinvested coupons will be greater

bond's price (equal to discounted value of remaining cash flows) will be smaller

when interest rates fall the two effects (called reinvestment risk and price risk) move in the opposite direction immunization property means that if the bond is valued at the duration date than the two effects exactly offset each other

proof:

- fair price of the bond

$$P = \frac{C}{1+r} + \ldots + \frac{C}{(1+r)^T} + \frac{M}{(1+r)^T} = \frac{C}{r}\left(1 - \frac{1}{(1+r)^T}\right) + \frac{M}{(1+r)^T}$$
$$= \frac{C}{r} - \frac{C}{r(1+r)^T} + \frac{M}{(1+r)^T}$$

- value of accumulated coupons at the duration date (reinvested coupon component)

$$\mathrm{RC} = C(1+r)^{D-1} + C(1+r)^{D-2} + \ldots + C(1+r) + C = C\frac{(1+r)^D - 1}{r}$$

- derivation with respect to r gives

$$\frac{d\mathrm{RC}}{dr} = \frac{C}{r}D(1+r)^{D-1} - \frac{C}{r^2}(1+r)^D + \frac{C}{r^2}$$

- value of the bond's price at the duration date (the capital gain component)

$$\mathrm{CG} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^{T-D}} + \frac{M}{(1+r)^{T-D}}$$
$$= \frac{C}{r}\left(1 - \frac{1}{(1+r)^{T-D}}\right) + \frac{M}{(1+r)^{T-D}}$$

- derivation with respect to r gives

$$\frac{d\mathrm{CG}}{dr} = -\frac{C}{r^2} + \frac{C}{r^2(1+r)^{T-D}} + \frac{C(T-D)}{r(1+r)^{T-D+1}} - \frac{M(T-D)}{r(1+r)^{T-D+1}}$$

- the assumption that reinvestment and price risks offset each other leads to the equation

$$0 = \frac{d\mathrm{RC}}{dr} + \frac{d\mathrm{CG}}{dr}$$
$$= D \times \left(\frac{C}{r} + \frac{M}{(1+r)^T} - \frac{C}{r(1+r)^T}\right) - C \times \left(\frac{1+r}{r^2} - \frac{1}{r^2(1+r)^{T-1}} - \frac{T}{r(1+r)^T}\right)$$
$$= D \times P - C \times \left(\frac{1+r}{r^2} - \frac{1}{r^2(1+r)^{T-1}} - \frac{T}{r(1+r)^T}\right)$$

- it can be shown that the above equation is consistent with the analytical formula for duration

f) duration of a portfolio of bonds is the weighted average of durations of individual bonds using relative market values of bonds as weights

$$
\begin{aligned}
D(B_1+B_2) &= \frac{n_1C_1+n_2C_2}{n_1P_1+n_2P_2}\sum_{t=1}^{T}\frac{t}{(1+r)^t}+\frac{n_1M_1+n_2M_2}{n_1P_1+n_2P_2}\times\frac{T}{(1+r)^T} \\
&= \frac{n_1P_1}{n_1P_1+n_2P_2}\left[\frac{C_1}{P_1}\sum_{t=1}^{T}\frac{t}{(1+r)^t}+\frac{M_1}{P_1}\times\frac{T}{(1+r)^T}\right] \\
&\quad+\frac{n_2P_2}{n_1P_1+n_2P_2}\left[\frac{C_2}{P_2}\sum_{t=1}^{T}\frac{t}{(1+r)^t}+\frac{M_2}{P_2}\times\frac{T}{(1+r)^T}\right] \\
&= \frac{n_1P_1}{n_1P_1+n_2P_2}D(B_1)+\frac{n_2P_2}{n_1P_1+n_2P_2}D(B_2)
\end{aligned}
$$

3.3 Convexity

convexity can be regarded as a second-order measure (quadratic approximation) of interest rate risk (as opposed to duration which can be regarded as a first-order measure (linear approximation of interest rate risk)

convexity measures the curvature of the present-value profile (as opposed to duration which measures the slope of the present-value profile)

fair price of a bond is a function $P = f(r)$

second-order Taylor expansion of the function gives

$$
\begin{aligned}
\Delta P &= \frac{dP}{dr}\times\Delta r+\frac{1}{2}\frac{d^2P}{dr^2}\times(\Delta r)^2+\dots \\
&\doteq -\frac{P}{1+r}\times D\times\Delta r+\frac{1}{2}\times P\times\left(\frac{1}{P}\times\frac{d^2P}{dr^2}\right)\times(\Delta r)^2
\end{aligned}
$$

convexity of the bond is called the expression

$$
K=\frac{1}{P}\times\frac{d^2P}{dr^2}=\frac{C}{P}\sum_{t=1}^{T}\frac{t(t+1)}{(1+r)^{t+2}}+\frac{M}{P}\times\frac{T(T+1)}{(1+r)^{T+2}}
$$

♦

Using the duration and convexity measures estimate a new price of the bond from the previous example if yields on five-year bonds are expected to rise from 8 to 10 %?

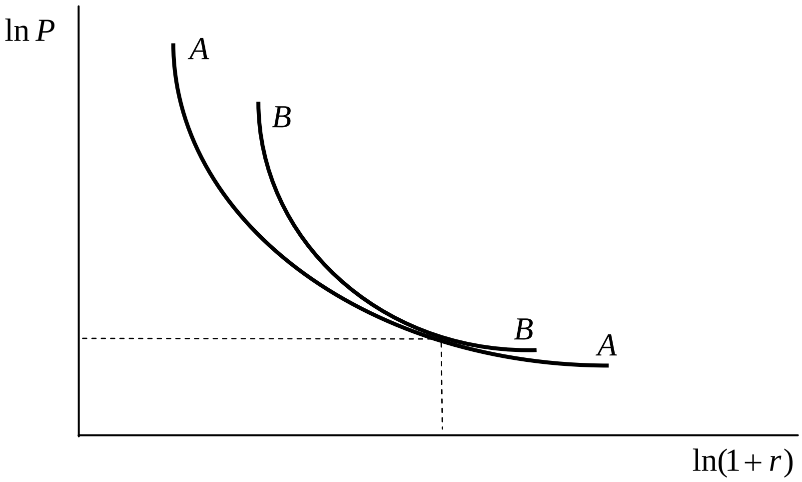
$$
\begin{aligned}
K &= \frac{5}{88.02}\times\left(\frac{1\times2}{1.08^3}+\frac{2\times3}{1.08^4}+\frac{3\times4}{1.08^5}+\frac{4\times5}{1.08^6}+\frac{5\times6}{1.08^7}\right)+\frac{100}{88.02}\times\frac{5\times6}{1.08^7} \\
&= 22.40
\end{aligned}
$$

$$\Delta P \doteq -7.35 + \frac{88.02}{2} \times 22.4 \times (0.1 - 0.08)^2 = -7.35 + 0.39 = -6.96$$

A bond price is expected to fall from 88.02 to (88.02 – 6.96) = 81.06. By comparing the estimated new price with the exact new price 81.05 we see that the convexity term substantially improved the estimation.

♦

Desirable property of convexity



two bonds *A* and *B* are trading at the same price, at the same yield to maturity and they have the same duration (they have the same slope of present-value profile)

bond *B* is more convex than the bond *A*

higher convexity is desirable because

- if yields rise then the price of *B* falls by less than the price of *A*
- if yields fall then the price of *B* rises by more than the price of *A*

this situation is not likely to happen because investors will be ready to pay to obtain higher convexity

the bond with the higher convexity would sell at a higher price and therefore with a lower yield

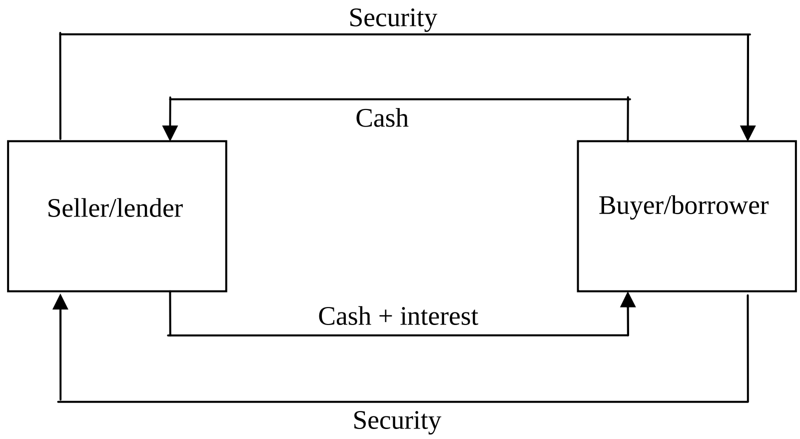
IV. REPO

4.1 Classical repo

repo is a short for *sale and repurchase agreement* whereby the two transactions involved are dealt as a package transacted under one agreement

the first deal is a sale of a security

the second deal is a reversal of the first deal



reverse repo is the opposite to the repo – initial purchase of securities followed by a subsequent sale (repo to one party is a reverse repo to the other party)

<u>legal vs. economic treatment of repo</u>

legal treatment: ownership of the security does pass from the security's seller to the security's buyer for the period of the repo

consequences: if the security's seller defaults on cash payment in the reversal deal then the counterparty can keep the security (the security is excluded from bankruptcy proceedings)

economic treatment: repo is a type of secured lending in which the cash lender holds the security only as collateral and is rewarded by being paid a regular interest (determined by a repo rate)

consequences: if a coupon is payable on the security during the repo term then the security's holder will receive it but he is obliged to make compensating payment to the security's seller similar deals

buy/sell-back is a deal in which the two legs of the classical repo, although dealt simultaneously, are treated as two separate transactions rather than one

security lending is a deal in which the lender of a special security lends the security for a fee and is secured against default by the borrower by taking collateral also in the form of securities

terminology

repo terminology is based on the securities side of the deal not on the cash side

lender, seller in the repo is the party who lends/sells securities at the outset and repurchases them later

borrower, buyer in the repo (called also investor) is the party who borrows/buys securities (and thus invests the cash) at the outset and sells them later

meaning of bid and offer rates: 5.20 – 5.30

money market

5.20 is the bid rate at which the dealer is willing to take deposit (he bids for cash)

5.30 is the offer rate at which the dealer is willing to make loan (he offers cash)

repo market

5.20 is the offer rate because the dealer offers securities against cash at 5.20 % (he effectively borrows cash at that rate)

5.30 is the bid rate because the dealer bids for securities against cash at 5.30 % (he effectively lends cash at that rate)

cash flows in the repo

cash-driven repo means that the deal is based on a particular amount of cash to be borrowed and the exact nature of the collateral is not important

security-driven repo means that the deal is initiated by the need to borrow a particular amount of security which is called the *special* (the more special the security, the lower the repo rate)

♦

A security-driven repo transaction is based on the following parameters:

Currency denomination: euro

Repo term: 28 days

Repo rate: 4.0 % (ACT/360 convention)

Special security: Treasury bond with a coupon of 8.5 %, actual clean price of 108.95 and accrued interest for 111 days

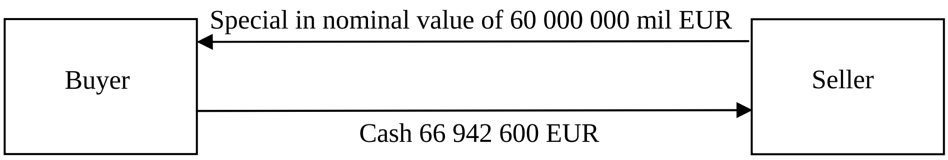
Required nominal value of the special: 60 mil EUR

Cash flow at the beginning of the repo deal:

accrued interest = $\frac{111}{360} \times 8.5 = 2.621\text{EUR}$

full price = $108.95 + 2.621 = 111.571$ EUR

cash paid = $60000000 \times \frac{111.571}{100} = 66942600$ EUR

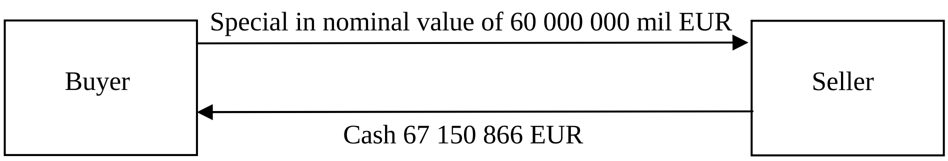


Cash flow at the end of the repo deal:

interest = $66942600 \times 0.04 \times \frac{28}{360} = 208266\,\text{EUR}$

principal = 66942600 EUR

cash paid = $208266 + 66942600 = 67150866$ EUR



♦

<u>margining</u>

a) initial margin has the form of a haircut

paid cash = market value of collateral × (1 ± haircut)

positive haircut (i.e. 2 %) ⇒ the collateral value is lower than the cash loan (the loan is under-collateralised)

negative haircut (i.e. -2 %) ⇒ the collateral value is higher than the cash loan (the loan is over-collateralised) the amount of haircut depends on the relative creditworthiness of the two parties

b) marking to market

the buyer recalculates the value of the collateral continually in order to ensure that it is of adequate value

if the value of the collateral has fallen the buyer may take a margin call requiring the seller to transfer more collateral (in terms of more securities or in cash)

if the value of the collateral rises the seller can make a margin call requiring the buyer to return some of the collateral

variation margin is the amount of collateral transferred between the seller and the buyer in response to a margin call

<u>substitution</u>

a repo deal may include the right for the security’s seller to change the exact security used as collateral during the period of the repo

<u>maturity</u>

the repo period is generally from one day (overnight repo) of several months

term repo – the period is fixed and agreed in advance

open repo – either party may call for the repo to be terminated at any time (requiring some notice), the repo is rolled over each day

<u>credit risk considerations</u>

double security of the repo structure is provided by

- credit standing of the counterparty
- credit standing of the issuer of the collateral

repo rates at which it is possible to borrow are lower then the rates for an unsecured interbank loan

<u>custody repo</u>

custody repo is a repo which involves another organisation to act as custodian for the collateral (a custodian holds the security in a separate account on the buyer’s behalf)

custodian performs a couple of duties (marking to market, daily reports, substitutions and others)

the buyer cannot to use the security in other repo deal

4.2 Applications of repo

i) <u>funding a long position in a bond</u>

motives: - dealer has bullish expectations and therefore wishes to purchase bonds

- dealer buys a bond which his counterparty intends to buy
- dealer covers a failed purchase when he intended to buy and sell the same bond

structure of a deal:

the dealer borrows cash for purchasing the bond through the repo trade in which the bond serves as the collateral

feasibility of the deal is ensured by the settlement process in which all payments of cash or transfers of a particular security are netted on any date across all the transactions outstanding between the dealer and the clearing system

ii) <u>covering a short position in a bond</u>

motives: - dealer has bearish expectations and therefore wishes to sell bonds short

- dealer sells a bond which his counterparty intends to sell
- dealer covers a failed sale when he intended to buy and sell the same bond

structure of a deal:

the dealer borrows a security in a reverse repo and obtains the cash collateral by selling the bond in the market

iii) <u>yield enhancement</u>

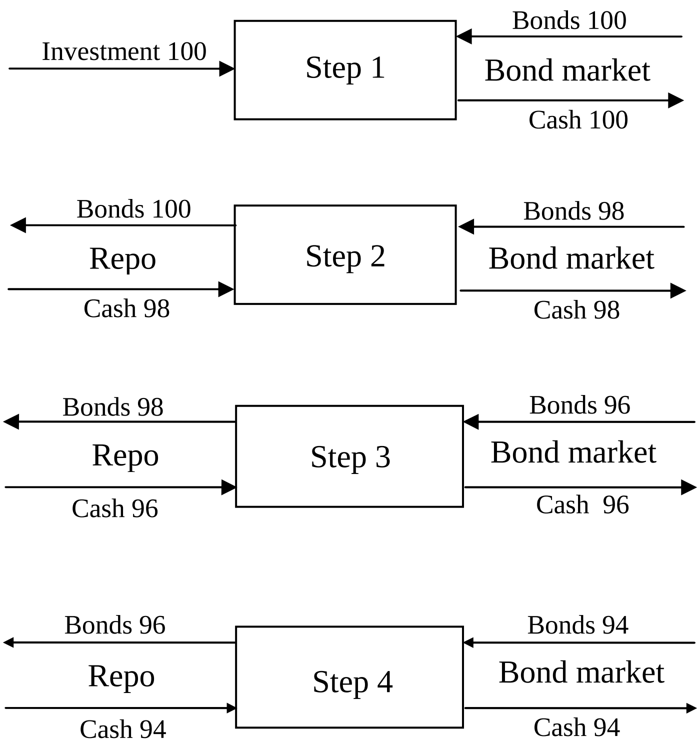
fund manager may sell a special security from his portfolio in a repo transaction and replace this special security by general collateral in a reverse repo transaction

both special and general collateral have the same creditworthiness so credit risk of the portfolio remains unchanged profit made is the difference between the two repo rates (the special allows to borrow at a lower rate in comparison with the lending rate

iv) <u>leveraging a bond portfolio</u>

existing holding of bonds can be used as collateral to repo in more cash that can be invested in new securities

the leveraging process can be repeated as far as prudence and margin requirements allow



original investment = 100 $

exposition in the bond market = 388 $

theoretical exposition at a 2 % haircut = 100 / 0.02 = 5000 $

if the bond market collapses margin calls will be made in each repo along the whole chain and heavy losses will result (fund manager is urged to sell bonds at depressed prices)

v) cash-and-carry transaction in long-term interest rate futures

process of delivering a CTD bond to the futures contract is based on a repo deal (in order to borrow the money to buy the bond the dealer obtains the bond thorough a repo transaction; he can repo the bond out)

implied repo rate is the break-even rate at which the cash-and-carry transaction gives a zero result (no profit and no loss)

vi) liquidity management

repo operations is a major tool of monetary policy conducted by central banks

central bank adds cash to the banking system by buying securities from the banks in open market operation called also *injecting repos*

central bank drains cash from the banking system by selling securities to the banks in open market operation called also *withdrawing repos*

repo rate at which repo transactions are conducted is the mechanism through which changes in interest rate levels are signalled to the market

V. MORTGAGE LOANS AND MORTGAGE-BACKED SECURITIES

mortgage market is a collection of markets that include a primary (origination) market where funds are borrowed for financing of housing and a secondary market in which mortgages trade

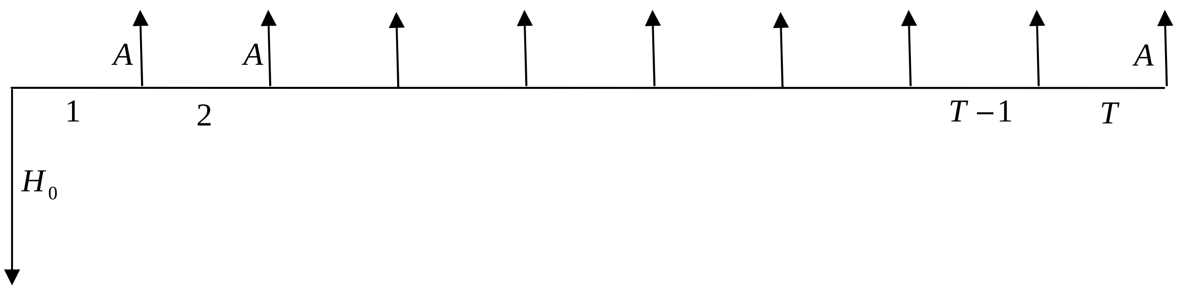
mortgage instruments are related to the primary mortgage market and represent various types of mortgage designs that intermediate housing financing

mortgage-backed securities (MBS) are related to the secondary mortgage market and represent various types of securities that derive its cash flow from an underlying pool of mortgages

securitization is a techniques of financial engineering in which individual assets are pooled and used as collateral for the issuance as well as source of cash flow of derivative securities

5.1 Traditional mortgage

traditional (level-payment, plain-vanilla) mortgage is a mortgage in which the borrower pays a fixed interest rate (the mortgage rate) and repays a part of the original mortgage balance (the principal) in equal instalments over an agreed period of time (term of the mortgage) in such a way that at the end of the term the loan has been fully repaid (the mortgage is fully amortized)



balloon mortgage is a mortgage in which payments are sufficient to pay only interest and no payments are made to reduce the original balance that must be paid at the end of the term of the mortgage (problem with refinancing the balance, disastrous consequences during the Great Depression) mathematics of a traditional mortgage

H_0 …original mortgage balance

H_t .. mortgage balance at the end of period t

A …regular mortgage payment (instalment)

T … term of mortgage

r … mortgage rate

fair value of a LPM is a present value of all future instalments discounted at a given mortgage rate

discounted present value should equal the original mortgage balance

$$H_0 = \sum_{t=1}^{T} \frac{A}{(1+r)^t} = A \times \frac{1}{r}\left(1 - \frac{1}{(1+r)^T}\right)$$

amount of a regular mortgage payment (annuity):

$$A = H_0 \times \frac{r(1+r)^T}{(1+r)^T - 1} = H_0 \times a_T \,, \qquad a_T \text{ is the annuity factor}$$

♦

A mortgage of 100000 $ was extended for 30 years at an annual mortgage rate of 9.5 %. The frequency of mortgage payments is monthly. Determine the value of the monthly mortgage payment.

$$H_0 = 100000,\ T = 30 \times 12 = 360,\ r = 0.095/12 = 0.0079167.$$

$$A = 100000 \times \frac{0.0079167 \times (1.0079167)^{360}}{(1.0079167)^{360} - 1} = 840.85\,\$$$

♦

decomposition of an instalment into interest payment and principal repayment

in each period a fixed mortgage payment consists of interest paid on an outstanding mortgage balance and repayment of a portion of the outstanding mortgage balance

$$A_t = A = r \times H_{t-1} + (H_{t-1} - H_t)$$

the formula for determining the remaining mortgage balance (at the end of the period t)

$$H_t = H_0 \frac{r(1+r)^T}{(1+r)^T - 1} + H_{t-1}(1+r) = H_0 \frac{(1+r)^T - (1+r)^t}{(1+r)^T - 1}$$

the formula for determining the interest payment (in the period *t*)

$$rH_{t-1} = H_0 \frac{r[(1+r)^T - (1+r)^{t-1}]}{(1+r)^T - 1}$$

the formula for determining the scheduled principal repayment (in the period *t*)

$$H_{t-1} - H_t = H_0 \frac{r(1+r)^{t-1}}{(1+r)^T - 1}$$

<u>repayment calendar of traditional mortgage loan</u>

calculation of the cash flow of an LPM can be arranged in a table called payment calendar

annuity amount approach:

1. Calculate the value of a regular fixed instalment using the annuity formula and insert the result into all cells in the column *Total instalment*
2. Calculate the interest payment (the number in the *Beginning balance* is multiplied by the mortgage rate)
3. Calculate the principal repayment (the number in *Interest payment* is subtracted from the number in *Total instalment*)
4. Calculate the ending balance (the number in *Principal repayment* is subtracted from the number in *Beginning balance*)
5. Use the number in *Ending balance* as the next period's number in *Beginning balance*
6. Repeat the sequence of steps from 2 through 5 for the subsequent period

♦

Make out the payment calendar for a mortgage with a principal of 200000 $, term to maturity of 20 years and an annual mortgage rate of 3 %. Frequency of instalment is yearly.

$$\text{total instalment} = 200000 \times \frac{0.03 \times (1.03)^{20}}{1.03^{20} - 1} = 13443.14$$

Year	Beginning balance	Regular instalment	Interest payment	Principal repayment	Unpaid balance
1	200000,00	13443,14	6000,00	7443,14	192556,86
2	192556,86	13443,14	5776,71	7666,44	184890,42
3	184890,42	13443,14	5546,71	7896,43	176993,99
4	176993,99	13443,14	5309,82	8133,32	168860,67
5	168860,67	13443,14	5065,82	8377,32	160483,35

6	160483,35	13443,14	4814,50	8628,64	151854,71
7	151854,71	13443,14	4555,64	8887,50	142967,21
8	142967,21	13443,14	4289,02	9154,13	133813,08
9	133813,08	13443,14	4014,39	9428,75	124384,34
10	124384,34	13443,14	3731,53	9711,61	114672,72
11	114672,72	13443,14	3440,18	10002,96	104669,76
12	104669,76	13443,14	3140,09	10303,05	94366,72
13	94366,72	13443,14	2831,00	10612,14	83754,58
14	83754,58	13443,14	2512,64	10930,50	72824,07
15	72824,07	13443,14	2184,72	11258,42	61565,65
16	61565,65	13443,14	1846,97	11596,17	49969,48
17	49969,48	13443,14	1499,08	11944,06	38025,42
18	38025,42	13443,14	1140,76	12302,38	25723,04
19	25723,04	13443,14	771,69	12671,45	13051,59
20	13051,59	13443,14	391,55	13051,59	0,00
Sum		268862,83	68862,83	200000,00	

♦

characteristic features of payment calendar:

- the portion of the regular payment applied to interest declines (because the mortgage balance is reduced with each instalment)
- the portion applied to reducing the mortgage balance increases (because the instalment is fixed and the interest component falls)
- the mortgage is completely paid off (self-amortized) at the end of the mortgage term

annuity factor approach:

a slightly modified approach to making out the payment calendar is based on the relationship:

$$A = H_0 \times a_T = H_1 \times a_{T-1} = \ldots = H_t \times a_{T-t} = H_T \times a_0$$

annuity factor approach can be interpreted as if the outstanding mortgage is prepaid at the end of each period by means of taking a new mortgage whose maturity is shorter by one period and whose principal equals the mortgage’s outstanding balance

sequence of steps:

1. Calculate the annuity factor for the beginning balance of the mortgage using the respective formula

2. Calculate the total instalment (the number in the *Beginning balance* is multiplied by the number in the *Annuity factor*)
3. Decompose the total instalment into an interest payment and principal repayment and calculate the ending balance in the same way as in the annuity level approach
4. Use the number in *Ending balance* as the next period's number in *Beginning balance*
5. Repeat the sequence of steps for the subsequent period

♦

Make out the payment calendar for the mortgage from the previous example (principal 200000 $, term to maturity 20 years, annual mortgage rate 3 %, yearly frequency of payment) using the annuity factor approach.

Year	Beginning balance	Annuity factor	Regular instalment	Interest payment	Principal repayment	Unpaid balance
1	200000,00	0,067216	13443,14	6000,00	7443,14	192556,86
2	192556,86	0,069814	13443,14	5776,71	7666,44	184890,42
3	184890,42	0,072709	13443,14	5546,71	7896,43	176993,99
4	176993,99	0,075953	13443,14	5309,82	8133,32	168860,67
5	168860,67	0,079611	13443,14	5065,82	8377,32	160483,35
6	160483,35	0,083767	13443,14	4814,50	8628,64	151854,71
7	151854,71	0,088526	13443,14	4555,64	8887,50	142967,21
8	142967,21	0,094030	13443,14	4289,02	9154,13	133813,08
9	133813,08	0,100462	13443,14	4014,39	9428,75	124384,34
10	124384,34	0,108077	13443,14	3731,53	9711,61	114672,72
11	114672,72	0,117231	13443,14	3440,18	10002,96	104669,76
12	104669,76	0,128434	13443,14	3140,09	10303,05	94366,72
13	94366,72	0,142456	13443,14	2831,00	10612,14	83754,58
14	83754,58	0,160506	13443,14	2512,64	10930,50	72824,07
15	72824,07	0,184598	13443,14	2184,72	11258,42	61565,65
16	61565,65	0,218355	13443,14	1846,97	11596,17	49969,48
17	49969,48	0,269027	13443,14	1499,08	11944,06	38025,42
18	38025,42	0,353530	13443,14	1140,76	12302,38	25723,04
19	25723,04	0,522611	13443,14	771,69	12671,45	13051,59
20	13051,59	1,030000	13443,14	391,55	13051,59	0,00
Sum			268862,83	68862,83	200000,00	

♦

5.2 Prepayments

three components of the mortgage cashflow:

- interest payment

- scheduled principal repayment
- ***prepayments*** are called payments in excess of the regularly scheduled principal

reasons for prepayments:

- selling the property (change of employment, purchase of a more expansive home, divorce, etc.)
- right of borrower to pay off all or part of the mortgage balance at any time (particularly if market rates fall below the rate in the contract)
- default on meeting mortgage obligation (the property is repossessed, sold and proceeds are used to pay off the mortgage, for insured mortgages the insurer will pay off the mortgage balance)
- insured catastrophe occurs (the insurance proceeds are used to pay off the mortgage)

prepayment rate is an estimate of a fraction of the remaining principal in the pool of mortgages that is prepaid each month

the rate is based on the historical prepayment experience and on the current and expected economic environment

$$\Pi_t = p_t \times (H_{t-1} - A_t)$$

Π_t … projected principal prepayment for period t

p_t … prepayment rate

H_{t-1} .. mortgage balance at the beginning of period t given all previous prepayments

A_t … scheduled principal repayment for period t

link between annual p_a and monthly p_m prepayment rates

$$p_m = 1 - \sqrt[12]{(1 - p_a)}$$

♦

Make out the payment calendar for the mortgage from the previous example (principal 200000 $, term to maturity 20 years, annual mortgage rate 3 %, yearly frequency of payment) assuming that the annual prepayment rate is 6 %.

Year	Beginning balance	Regular instalment	Interest payment	Principal repayment	Prepay-ments	Unpaid balance
1	200000,00	13443,14	6000,00	7443,14	11553,41	181003,45
2	181003,45	13443,14	5430,10	8013,04	10379,42	162610,98
3	162610,98	13443,14	4878,33	8564,81	9242,77	144803,40
4	144803,40	13443,14	4344,10	9099,04	8142,262	127562,10
5	127562,10	13443,14	3826,86	9616,28	7076,749	110869,07
6	110869,07	13443,14	3326,07	10117,07	6045,12	94706,88
7	94706,88	13443,14	2841,21	10601,94	5046,297	79058,65
8	79058,65	13443,14	2371,76	11071,38	4079,236	63908,03
9	63908,03	13443,14	1917,24	11525,90	3142,928	49239,20
10	49239,20	13443,14	1477,18	11965,97	2236,394	35036,85
11	35036,85	13443,14	1051,11	12392,04	1358,689	21286,12
12	21286,12	13443,14	638,58	12804,56	508,8937	7972,67
13	7972,67	8211,85	239,18	7972,67	0	0
14	0	0	0	0	0	0
15	0	0	0	0	0	0
16	0	0	0	0	0	0
17	0	0	0	0	0	0
18	0	0	0	0	0	0
19	0	0	0	0	0	0
20	0	0	0	0	0	0
Sum		169529,55	38341,72	131187,82	68812,18	

The 6 % prepayment rate shortened the life of the mortgage to only 13 years.

The sum of scheduled principal repayments and prepayments equals 200000 $.

♦

prepayment risk is the risk that the homeowner will prepay the mortgage at any unfavourable time and that the timing and amount of prepayments are not known with certainty and are difficult to predict accurately

contraction risk is related to the situation when declining mortgage rates tend to speed up prepayments (because homeowners have an incentive to refinance their debts at a lower rate) and investors are forced to reinvest prepayments at lower market interest rates

extension risk is related to the situation when rising mortgage rates tend to slow down prepayments (because homeowners lose an incentive to refinance their debts at a lower rate) and investors cannot reinvest more cash at higher market interest rates

mismatch risk occurs when speeding up and slowing down prepayments disturb the balanced position between assets and liabilities in terms of their durations

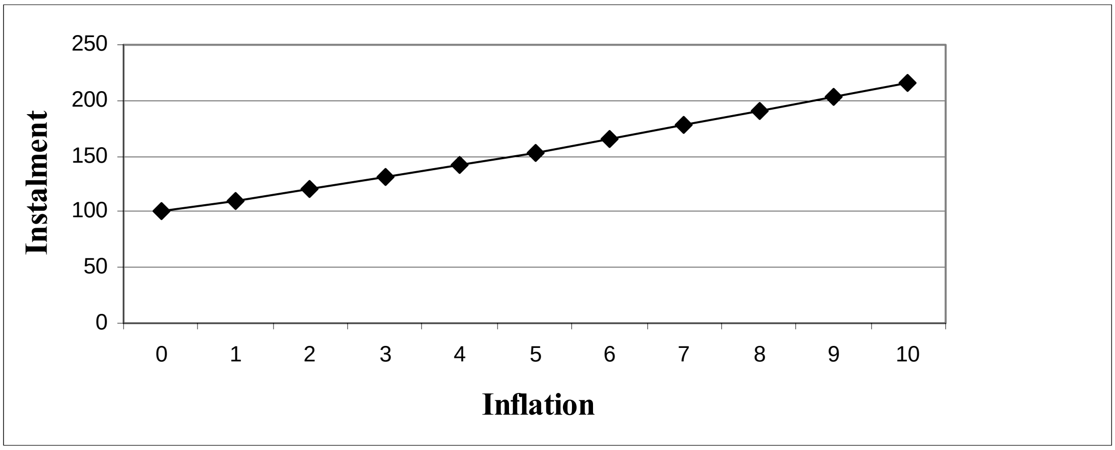
5.3 Inflation-adjusted mortgage designs

problems caused by high inflation in the traditional level-payment mortgage:

erosion problem: substantial inflation erodes the repayments of principal in terms of purchasing power (i.e. by the twentieth year the real value of the initial instalment will be down to some 15 % at a 10 % inflation per year)

if the erosion is not made up by sufficiently high mortgage rates that preserve required real interest rates then lenders will be discouraged to provide adequate funding for the mortgage market

Nature of the tilt problem



tilt problem: if high inflation is addressed by high mortgage rates in order to leave real rates largely unchanged then the level-payment mortgage may lead to high initial instalments

high initial payments tend to foreclose home ownership to large segments of the population or force borrowers to scale down their demands

a) graduated-payment mortgage

GPM plan consists in replacing the level-payment stream with one that grows at the rate of inflation in such way that the graduated stream has the same present value as the underlying level-payment stream

ρ … required real rate of interest (mortgage rate in the absence of inflation)

π … rate of inflation

r … mortgage rate, $r = (1+\rho)\times(1+\pi)-1$

G_0 .. initial value of graduated payment, $\quad G_t = G_0(1+\pi)^t$

initial value G_0 can be found from the equation

$$H_0 = \sum_{t=1}^{T} \frac{G_0 \times (1+\pi)^t}{(1+r)^t} = \sum_{t=1}^{T} \frac{G_0 \times (1+\pi)^t}{(1+\rho)^t \times (1+\pi)^t} = G_0 \sum_{t=1}^{T} \frac{1}{(1+\rho)^t}$$

$\Rightarrow$ G_0 is the instalment that would be paid in the absence of inflation

♦

Compare payment calendars between a 20-year level-payment mortgage and a 20-year graduated-payment mortgage with beginning balances of 200000 $ under assumption that a mortgage bank requires a real interest rate of 3 % and there is a steady inflation of 8 %.

Payment calendar for the level-payment mortgage

mortgage rate = $(1+0.03)\times(1+0.08)-1 = 0.1124$

level payment = $200000 \times \dfrac{0.1124 \times 1.1124^{20}}{1.1124^{20}-1} = 25510.38$

Year	Beginning balance	Regular instalment	Interest payment	Principal repayment	Unpaid balance
1	200000,00	25510,38	22480,00	3030,38	196969,62
2	196969,62	25510,38	22139,39	3371,00	193598,62
3	193598,62	25510,38	21760,49	3749,90	189848,72
4	189848,72	25510,38	21339,00	4171,39	185677,34
5	185677,34	25510,38	20870,13	4640,25	181037,09
6	181037,09	25510,38	20348,57	5161,81	175875,28
7	175875,28	25510,38	19768,38	5742,00	170133,28
8	170133,28	25510,38	19122,98	6387,40	163745,88
9	163745,88	25510,38	18405,04	7105,35	156640,53
10	156640,53	25510,38	17606,40	7903,99	148736,54
11	148736,54	25510,38	16717,99	8792,39	139944,15
12	139944,15	25510,38	15729,72	9780,66	130163,49
13	130163,49	25510,38	14630,38	10880,01	119283,48
14	119283,48	25510,38	13407,46	12102,92	107180,57
15	107180,57	25510,38	12047,10	13463,29	93717,28
16	93717,28	25510,38	10533,82	14976,56	78740,72
17	78740,72	25510,38	8850,46	16659,92	62080,80
18	62080,80	25510,38	6977,88	18532,50	43548,30
19	43548,30	25510,38	4894,83	20615,55	22932,74
20	22932,74	25510,38	2577,64	22932,74	0,00
Sum		510207,64	310207,64	200000,00	

Payment calendar for the graduated-payment mortgage (growing at 8 % per year)

mortgage rate = $(1+0.03)\times(1+0.08)-1=0.1124$

initial value of graduated payment = $200000\times\dfrac{0.03\times1.03^{20}}{1.03^{20}-1}=13443.14$

Year	Beginning balance	Regular instalment	Interest payment	Principal repayment	Unpaid balance
1	200000,00	14518,59	22480,00	-7961,41	207961,41
2	207961,41	15680,08	23374,86	-7694,78	215656,19
3	215656,19	16934,49	24239,76	-7305,27	222961,46
4	222961,46	18289,25	25060,87	-6771,62	229733,08
5	229733,08	19752,39	25822,00	-6069,61	235802,69
6	235802,69	21332,58	26504,22	-5171,65	240974,34
7	240974,34	23039,18	27085,52	-4046,33	245020,67
8	245020,67	24882,32	27540,32	-2658,01	247678,68
9	247678,68	26872,90	27839,08	-966,18	248644,86
10	248644,86	29022,73	27947,68	1075,05	247569,81
11	247569,81	31344,55	27826,85	3517,71	244052,10
12	244052,10	33852,12	27431,46	6420,66	237631,44
13	237631,44	36560,29	26709,77	9850,51	227780,93
14	227780,93	39485,11	25602,58	13882,53	213898,40
15	213898,40	42643,92	24042,18	18601,74	195296,66
16	195296,66	46055,43	21951,34	24104,09	171192,57
17	171192,57	49739,87	19242,05	30497,82	140694,75
18	140694,75	53719,06	15814,09	37904,97	102789,79
19	102789,79	58016,58	11553,57	46463,01	56326,78
20	56326,78	62657,91	6331,13	56326,78	0,00
Sum		664399,33	0,00	200000,00	

♦

characteristic features of GPM payment calendar:

- in comparison with the LPM the GPM instalments are substantially lower at the beginning of the mortgage life but they are substantially higher at the end of the mortgage life

  problem for people whose income may not keep up with inflation (the last instalment is over 4.3 times larger than the first instalment)

  most GPM plans therefore use modified versions in which the nominal payment grows during a portion of the life of the contract and thereafter levels off

- *negative amortization* is a situation in which the periodic payment is insufficient to cover accrued interest and thus the outstanding mortgage balance increases the debtor instead of repaying is getting deeper and deeper in debt as a consequence of high and persistent inflation

negative amortization may raise the mortgage balance to an amount that exceeds the market value of the property encouraging the borrower to default

b) price-level-adjusted mortgage

PLAM plan is designed to be fixed in purchasing power terms and the effective instalment and the unpaid balance are adjusted by an inflation correction factor

terms of the contract:

- the real interest rate (the mortgage rate that could be expected if there were no inflation in the economy, about 3-4 %)
- the inflation correction factor (equal to an increase in the selected price index in a given period)
- other terms of the loan (maturity, initial amount, etc.)

sequence of steps:

1. Work out the payment calendar for an underlying real mortgage interest rate (using the annuity amount approach)
2. Multiply the real instalment (prevailing in the absence of inflation) by an appropriate inflation correction factor
3. Multiply the real unpaid balance by the same correction factor

♦

Make out the payment calendar for a PLAM plan with a principal of 200000 $ and term to maturity of 20 years under assumption that the bank charges a 3 % real mortgage rate. Persistent consumer inflation is expected to stay at a level
o 8 % per year.

Year	Real beginning balance	Real instalment	Real unpaid balance	Price index	Regular instalment	Effective unpaid balance	Principal repayment
1	200000,00	13443,14	192556,86	1,0800	14518,59	207961,41	-7961,41
2	192556,86	13443,14	184890,42	1,1664	15680,08	215656,19	-7694,78
3	184890,42	13443,14	176993,99	1,2597	16934,49	222961,46	-7305,27

4	176993,99	13443,14	168860,67	1,3605	18289,25	229733,08	-6771,62
5	168860,67	13443,14	160483,35	1,4693	19752,39	235802,69	-6069,61
6	160483,35	13443,14	151854,71	1,5869	21332,58	240974,34	-5171,65
7	151854,71	13443,14	142967,21	1,7138	23039,18	245020,67	-4046,33
8	142967,21	13443,14	133813,08	1,8509	24882,32	247678,68	-2658,01
9	133813,08	13443,14	124384,34	1,9990	26872,90	248644,86	-966,18
10	124384,34	13443,14	114672,72	2,1589	29022,73	247569,81	1075,05
11	114672,72	13443,14	104669,76	2,3316	31344,55	244052,10	3517,71
12	104669,76	13443,14	94366,72	2,5182	33852,12	237631,44	6420,66
13	94366,72	13443,14	83754,58	2,7196	36560,29	227780,93	9850,51
14	83754,58	13443,14	72824,07	2,9372	39485,11	213898,40	13882,53
15	72824,07	13443,14	61565,65	3,1722	42643,92	195296,66	18601,74
16	61565,65	13443,14	49969,48	3,4259	46055,43	171192,57	24104,09
17	49969,48	13443,14	38025,42	3,7000	49739,87	140694,75	30497,82
18	38025,42	13443,14	25723,04	3,9960	53719,06	102789,79	37904,97
19	25723,04	13443,14	13051,59	4,3157	58016,58	56326,78	46463,01
20	13051,59	13443,14	0,00	4,6610	62657,91	0,00	56326,78
Sum		268862,83			664399,33		200000,00

♦

characteristic features of PLAM payment calendar:

- effect of negative amortization means that the ending balance may keep growing for a substantial part of the life of the mortgage
- effect of growing amount of instalment means that payments start low and then arise smoothly at the rate of inflation (in this case the stream is equal to the GPM plan, in reality the inflation correction factor mirrors the actual development of price index)
- the mortgage loan is fully amortized by its construction (because it is fully amortized in real terms)

c) dual-rate mortgage

DRM plan combines a fixed rate which determines how much the borrower will effectively pay in regular instalments and a floating rate which determines effective return on a mortgage loan

terms of the contract:

- the payment rate which is a rate of interest fixed for the life of the mortgage loan
- the effective rate which changes periodically on the basis of an agreed-upon reference short-term rate
- other terms of the loan (maturity, initial amount, etc.)

sequence of steps:

1. On the basis of the fixed payment rate calculate the amount of instalment using the annuity factor approach
2. On the basis of the variable rate calculate the amount the interest component in the instalment
3. Calculate the amount of the repaid principal and the amount of unpaid balance
4. Use the amount of unpaid balance as the amount of subsequent period's beginning balance
5. Repeat the sequence of steps for the subsequent period

♦

Make out the payment calendar for a DRM plan with a principal of 200000 $ and term to maturity of 20 years under assumption that the bank charges a 3 % payment rate (equal to the real rate prevailing in the absence of inflation) and a 11 % effective rate (corresponding to a nominal rate composed of a 3 % real rate and an 8 % steady inflation rate).

Year	Beginning balance	Annuity factor	Regular instalment	Effective rate	Interest payment	Principal repayment	Unpaid balance
1	200000,00	0,067216	13443,14	0,11	22000,00	-8556,86	208556,86
2	208556,86	0,069814	14560,16	0,11	22941,25	-8381,09	216937,95
3	216937,95	0,072709	15773,28	0,11	23863,17	-8089,90	225027,85
4	225027,85	0,075953	17091,43	0,11	24753,06	-7661,63	232689,48
5	232689,48	0,079611	18524,61	0,11	25595,84	-7071,24	239760,71
6	239760,71	0,083767	20083,94	0,11	26373,68	-6289,74	246050,46
7	246050,46	0,088526	21781,95	0,11	27065,55	-5283,60	251334,06
8	251334,06	0,094030	23632,83	0,11	27646,75	-4013,92	255347,98
9	255347,98	0,100462	25652,79	0,11	28088,28	-2435,49	257783,47
10	257783,47	0,108077	27860,58	0,11	28356,18	-495,60	258279,07
11	258279,07	0,117231	30278,19	0,11	28410,70	1867,49	256411,58
12	256411,58	0,128434	32931,93	0,11	28205,27	4726,65	251684,93
13	251684,93	0,142456	35854,13	0,11	27685,34	8168,78	243516,14
14	243516,14	0,160506	39085,89	0,11	26786,78	12299,11	231217,03
15	231217,03	0,184598	42682,09	0,11	25433,87	17248,21	213968,82
16	213968,82	0,218355	46721,07	0,11	23536,57	23184,50	190784,32
17	190784,32	0,269027	51326,14	0,11	20986,27	30339,87	160444,45
18	160444,45	0,353530	56721,99	0,11	17648,89	39073,10	121371,36
19	121371,36	0,522611	63429,99	0,11	13350,85	50079,14	71292,22
20	71292,22	1,030000	73430,99	0,11	7842,14	65588,84	5703,38
Sum			670867,08		476570,46	194296,62	

The final instalment of 73430.99 $ must be topped up by the remaining value of unpaid balance of 5703.38 $. This additional payment makes the total sum of principal repayments equal to 200000 $.

♦

5.4 Mismatch-adjusted mortgage designs

mismatch problem arises when mortgages being very long-term assets are largely financed by funds of a short-term nature (the result is highly speculative activity of borrowing short and lending very long)

in the presence of rising interest rates lending institutions may become technically insolvent because the market value of its assets (discounted at the current high rates) will be insufficient to cover its liabilities

the mismatch problem can by solved by redesigning the traditional mortgage to produce an asset that would match the short-term market rates and thus the cost of liabilities

adjustable rate mortgage is a contract in which the mortgage rate is reset periodically in accordance with some appropriately chosen benchmark index

at the reset date the new contract rate is equal to the benchmark index plus a spread

possible resetting dates: every month, six months, one year, two years, five years

possible benchmark indexes:

- market-determined short-term interest rate
- cost of funding of mortgage originators (weighted average interest rate cost paid by originators for their liabilities)

ARM design does not address the tilt problem because the critical early payments will be roughly as high as they are with a traditional mortgage and will make a substantial jump every time the mortgage rate is adjusted

<u>caps in ARMs</u>

caps are limits that are meant to protect the borrower against sharp unfavourable changes in payment conditions at reset dates

periodic rate cap limits the change in the contract rate at the reset date (i.e. the rate may not increase by more then 2 % per annum)

periodic payment cap limits the change in the monthly instalment at the reset date

lifetime cap is an upper limit on the contract rate that could be charged over the life of the loan and is expressed in terms of the initial rate (i.e. if the initial mortgage rate is 7 % and the lifetime cap is 5 % then the maximum interest rate the lender can charge over the life of the loan is 12 %)

cap is an option that the lender has effectively sold to the homeowner

some of the expected loss for mortgage originator or investors due to caps will be recouped by a mortgage rates or other less favourable terms

<u>floors in ARMs</u>

floors are lower limits on changes in interest rate or payment amount at reset days

floors can be of periodic as well as lifetime nature

floor is an option that the homeowner has effectively sold to the lender

hybrid mortgage is a contract that shares features of fixed-rate and adjustable-rate mortgages

convertible adjustable rate mortgage is such ARM that can be converted into a fixed-rate mortgage

convertible (reducible) fixed rate mortgage can have the mortgage rate lowered if rates fall by some predetermined level

the borrower has the choice of converting and the lender charges a nominal fee for conversion

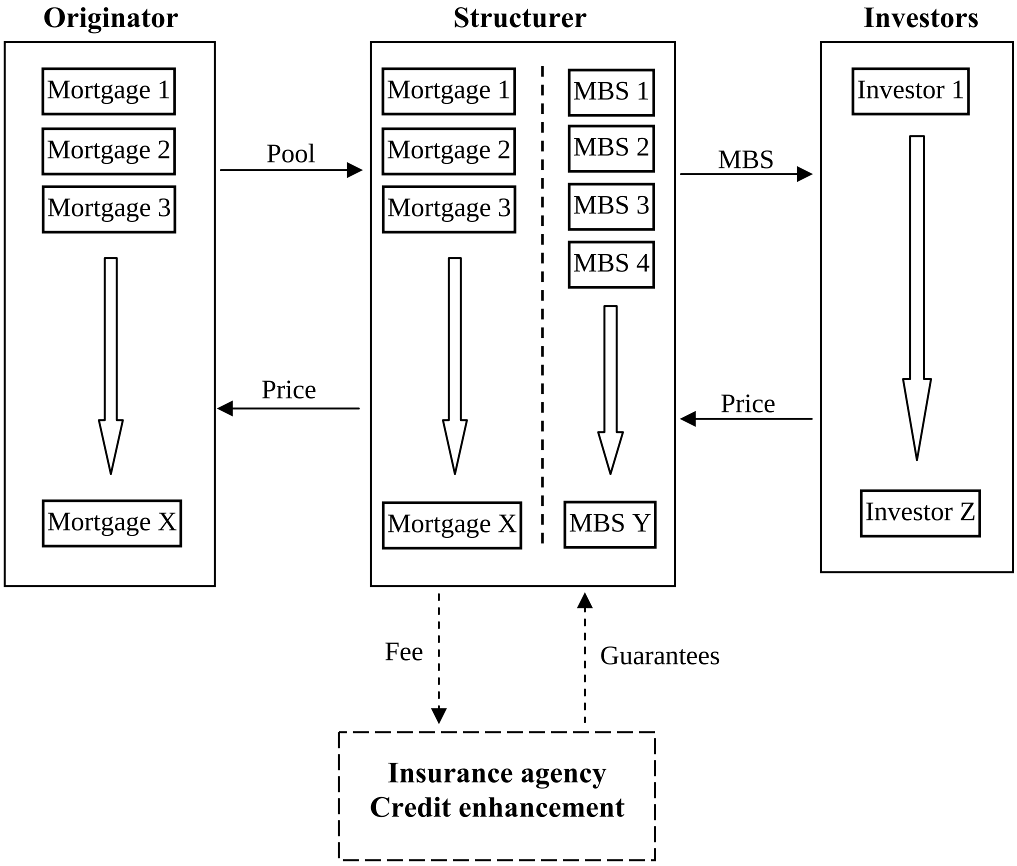
balloon (reset) mortgage is a contract in which the borrower is given long-term financing but at specified future dates the contract rate is renegotiated

the contract is effectively a short-term loan in which the lender agrees to provide financing for the remainder of the term of the mortgage

5.5 Mortgage-backed securities

mortgage securitization comprises various techniques of financial engineering in which individual mortgages are pooled and used as source of cash flow (collateral) for the creation of new securities called ***mortgage-backed securities*** **(*MBS*)**

Building blocks of securitization in the mortgage market



a) passthrough securities

passthrough security is a financial instrument that redirects the cashflow from an underlying pool of securitized mortgages to investors *on a pro rata basis*

advantages of passthroughs:

- by holding passthroughs the investor holds a diversified portfolio of mortgages that reduces most unsystematic risk and leaves only systemic risk
- liquidity of passthroughs is considerably better than that of an individual mortgage (by selling a passthrough the investor can dispose of all loans simultaneously, there is no need to dispose of loans one by one)

♦

An investment bank purchased a pool of 500 mortgage loans each of it in an amount of 100000 $. Against this pool the bank issued 5000 passthroughs. A 1 million dollar investment can thus buy:

a) 10 individual mortgages [1000000:100000] giving the right to a cash flow from these mortgages (substantial unsystematic risk)

b) 100 passthroughs [1000000:(500×100000:5000)] giving the right to 2 % [100:5000] of total cash flow from the pool of all 500 mortgages (well diversified portfolio)

♦

agency passthroughs are passthrough securities issued or guaranteed by government or government-sponsored institutions

USA: Federal National Mortgage Association (Fannie Mae)

Federal Home Loan Mortgage Corporation (Freddie Mac)

Government National Mortgage Association (Ginnie Mae))

conventional (private label) passthroughs are issued by private conduits without any government guarantees; they are rated by commercial rating agencies

forms of credit enhancement:

- corporate guarantees (the issuer uses its own credit rating to back the security)
- letter of credit (a guarantee provided by a financial institution)
- insurance (policy provided by an insurance company)
- over-collateralization (projected cash flow from un underlying mortgage pool exceeds liabilities from MBS)
- senior/subordinated structures (mortgage pool is partitioned into senior and subordinated part, senior part is sold to investors as conventional passthroughs and subordinated part can be either retained or sold to investors willing to accept the greater default risk, the amount of subordinated part to senior part determines credit rating of passthroughs)

b) collateralized mortgage obligations (CMO)

CMOs are bonds backed by the cash flow from an underlying pool of mortgages or an underlying pool of passthroughs (CMOs backed by other CMOs are called CMO square)

a CMO belongs to a CMO class (tranche) within a given CMO structure for which a given set of rules indicates how the principal from the underlying pool is to be distributed among the holders of bonds of individual CMO classes (i.e. sequential-pay, PAC bonds, accrual bonds, floating-rate bonds)

c) stripped mortgage-backed securities

strips are created by altering the distribution of principal and interest from a pro rata distribution to an unequal distribution

IO (interest only) class absorbs all of the interest from an underlying pool

PO (principal only) class absorbs the entire principal from an underlying pool

<u>properties of IOs and POs</u>

price of a PO moves in the opposite direction as the change in interest rates (a PO has a positive duration)

- mortgage rates fall ⇨ prepayments speed up and PO holders recover more quickly a given return (equal to nominal amount of a mortgage pool minus purchasing PO price)

this effect is strengthened by the cashflow discounted at a lower discount rate

- mortgage rates rise ⇨ prepayments slow down and it is taking longer to recover principal repayments

this effect is strengthened by the cashflow discounted at a higher discount rate

price of an IO tend to move in the same direction as the change in interest rates (an IO has a negative duration)

- mortgage rates decline ⇨ prepayments accelerate, outstanding principal declines faster thereby lowering the base for calculating the interest

this effect is usually stronger then the offsetting effect of the cashflow discounted at a lower discount rate

- mortgage rates increase ⇨ prepayments decelerate that improves the base for calculating the interest

this effect is usually stronger then the offsetting effect of the cashflow discounted at a higher discount rate

VI. MONEY MARKET SECURITIES

money market securities are short-term financial instruments with maturities of less than a year

capital market securities have maturities in excess of one year

6.1 Money market conventions

a) <u>day-count conventions</u>

as a general rule the calculation of interest takes account of the exact number of days between payments

variations arise in the conventions used for the number of days in the base year

ACT/365 … assumes a 365-day year

ACT/360 … assumes a 360-day year

♦

The yield on a deposit quoted on ACT/360 basis is 10.5 %. What is the equivalent yield expressed on ACT/365 basis?

$$r_{ACT/365} = r_{ACT/360} \times \frac{365}{360} = 10.5 \times 1.014 = 10.65\ \%$$

an amount of 100 \$ deposited for 20 days at $r_{ACT/360}$ yields

$$R = 100 \times \left(1 + r_{ACT/360} \times \frac{20}{360}\right) = 100 \times \left(1 + 0.105 \times \frac{20}{360}\right) = 100.58$$

an amount of 100 \$ deposited for 20 days at $r_{ACT/365}$ yields

$$R = 100 \times \left(1 + r_{ACT/365} \times \frac{20}{365}\right) = 100 \times \left(1 + 0.1065 \times \frac{20}{365}\right) = 100.58$$

♦

b) <u>reference rates</u>

offer rate is the rate at which one bank can temporarily lend funds to another bank

LIBOR (London Interbank Offer Rate) is a set of offer rates for various terms determined in the London interbank market

LIBOR rates are quoted in a number of major currencies and for a number of different terms (ranging from overnight to one year)

many money market interest rates are set with reference to LIBOR rates similar rates in other money markets: EURIBOR (Eurozone), PRIBOR (Prague), NYBOR (New York), TIBOR (Tokyo), etc.

bid rate is the rate at which one bank accepts funds from another bank

LIBID, PRIBID, NYBID, TIBID and so on are bid rates that are counterparts to offer rates in respective national money markets

c) <u>types of quotations</u>

quotation on a yield basis

on the maturity day the investor receives the nominal value of the instrument plus an amount of interest calculated according to the formula for simple interest

M … nominal (par) value of the instrument

r … quoted interest rate

T … number of days to maturity

interest = $M \times r \times \frac{T}{365}$

maturity value = $M \times \left(1 + \frac{T}{365} r\right)$

effective rate of interest is the annualized interest rate which would give the same compound return as the rate we are comparing

effective rate = $e = \left(1 + r\frac{T}{365}\right)^{365/T} - 1$

quotation on a discount basis

on the maturity day the investor receives the par value and no explicit interest which is however reflected in the difference between the discounted issue price and the par value

M … nominal (par) value of the instrument

d … quoted interest rate

T … number of days to maturity

discount = $M \times d \times \frac{T}{365}$

issuing price = $M \times \left(1 - \frac{T}{365} d\right)$

equivalent yield = $q = \left(\frac{M}{M - Md\frac{T}{365}} - 1\right) \times \frac{365}{T} = \frac{d}{1 - d\frac{T}{365}}$

the equivalent yield is always greater than the discount rate

$$d = q \times \left(1 - d\,\frac{T}{365}\right) < q \times 1 = q$$

c) <u>implied forward money market rates</u>

interest parity equation:

$$\left(1 + r_{T1}\,\frac{T_1}{365}\right) \times \left(1 + {}_{T1}f_{T2}\,\frac{T_2 - T_1}{365}\right) = \left(1 + r_{T2}\,\frac{T_2}{365}\right)$$

approximate formula for implied forward rates (neglecting the term with the product of rates)

$${}_{T1}f_{T2} \doteq \frac{r_{T2} \times T_2 - r_{T1} \times T_1}{T_2 - T_1}$$

standard property of a money market yield curve for expected changes in interest rates

upward sloping yield curve indicates that the market expects future interest rates to rise

$$r_{T1} < r_{T2} \quad \Rightarrow \quad r_{T1} < {}_{T1}f_{T2}$$

downward sloping yield curve indicates that the market expects future interest rates to fall

$$r_{T1} > r_{T2} \quad \Rightarrow \quad r_{T1} > {}_{T1}f_{T2}$$

6.2 Examples of money market instruments

a) money market deposit is an instrument which pays a fixed interest (quoted on yield basis) and has a fixed term

the interest and principal are paid in one lump sum on the maturity day

the deposit cannot be liquidated before maturity

♦

A money market deposit of 1 mil USD has been opened on 15 January at an interest rate of 9.25 % for one month.

maturity value = $1000000 \times \left(1 + 0.0925 \times \frac{31}{365}\right) = 1007856.16$

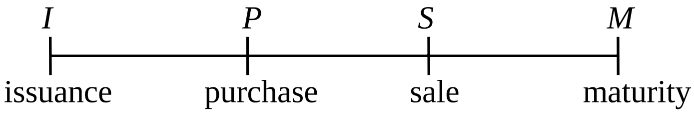
effective rate of interest = $\left(1 + 0.0925 \times \frac{31}{365}\right)^{365/31} - 1 = 0.0965 = 9.65\ \%$

♦

b) negotiable certificate of deposit carries a fixed interest (quoted on yield basis) and has a fixed term

CDs cannot be withdrawn before maturity but they can be traded in a secondary market (they are negotiable)

some CDS have maturity in excess of one year



$$\text{purchasing price} = P_P = \frac{M \times \left(1 + c \times \frac{T_{IM}}{365}\right)}{1 + r_P \times \frac{T_{PM}}{365}}$$

$$\text{selling price} = P_S = \frac{M \times \left(1 + c \times \frac{T_{IM}}{365}\right)}{1 + r_S \times \frac{T_{SM}}{365}}$$

holding period yield

$$r_h = \frac{P_S - P_P}{P_P} \times \frac{365}{T_{PS}} = \left[\frac{1 + r_P \times \frac{T_{PM}}{365}}{1 + r_S \times \frac{T_{SM}}{365}} - 1\right] \times \frac{365}{T_{PS}}$$

c … coupon rate

r_P, r_S … yield on the CD at purchase or at sale

T_{IM} … number of days between issue and maturity

T_{PM}, T_{SM} … number of days between purchase and maturity or sale and maturity

♦

Consider an investor who purchases a 91-day CD (with a coupon of 10 % and nominal value of 100) with 50 days to maturity at a yield of 10 % and sells 30 days later at a yield of 10 %.

$$\text{purchasing price} = P_P = \frac{100 \times \left(1 + 0.1 \times \frac{91}{365}\right)}{1 + 0.1 \times \frac{50}{365}} = 101.108$$

selling price = $P_S = \dfrac{100 \times \left(1 + 0.1 \times \dfrac{91}{365}\right)}{1 + 0.1 \times \dfrac{20}{365}} = 101.935$

holding period yield = $r_h = = \left[\dfrac{1 + 0.1 \times \dfrac{50}{365}}{1 + r_S \times \dfrac{20}{365}} - 1\right] \times \dfrac{365}{30} = 0.0995 = 9.95\ \%$

Even though the yield has not changed from 10 % over the period, the holding period return is less than 10 %. This is the result of compounding: the CD is priced to include accrued interest but the interest is not paid until maturity.

♦

c) **Treasury bill** is a short-term government security sold on the basis of a discount to the par value

on maturity the holder receives the par value of the bill

♦

A 91-day 100 $ Treasury bill is issued at a discount of 10 %.

issue price = $100 \times \left(1 - 0.1 \times \dfrac{91}{365}\right) = 97.51$

equivalent yield = $q = \dfrac{0.1}{1 - 0.1 \times \dfrac{91}{365}} = 0.1026 = 10.26\ \%$

♦

<u>other securities quoted on a discount basis</u>

bills of exchange (commercial bills) are issued by private companies against the sale of goods; they are used to finance trade in the short term

banker's acceptance is a written promise issued by the borrower to the bank to repay borrowed funds; it can be negotiable and can be sold in a secondary market; if the borrower defaults the investor has legal recourse to the bank that made the first acceptance

commercial paper refers to unsecured promissory notes issued by large corporations; the notes are not backed by any collateral but they rely on the high credit rating of the issuing corporation