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Intermediate macroeconomics

Exercises for topic of economic growth

Exercise 1: Production function

1. **Meaning of properties.** Consider the properties of neoclassical production function . List all the properties from lecture. For each property discuss whether it is reasonable or not, and come up with (real or fictitious) counterexample. (Hint: For one property think of fully autonomous robots.)

2. **Cobb-Douglas production function.** Consider the Cobb-Douglas production function $Y = AK^\alpha L^{1-\alpha}$. Show that it satisfies all the properties of neoclassical production function.

3. **Returns to scale.** Prove that each of the above production functions exhibits *constant returns to scale.*

   (a) **Linear production function**:
   $$Y = A \cdot (L + K)$$

   (b) **Leontief production function** (also known as the fixed-proportions production function):
   $$Y = \min\left(\frac{L}{a}, \frac{K}{b}\right)$$
   where a and b are constants that represent the fixed input coefficients for labor and capital, respectively.

   (c) **Constant Elasticity of Substitution (CES) production function**:
   $$Y = A\left(\alpha L^\rho + (1-\alpha)K^\rho\right)^{\frac{1}{\rho}}$$
   where $\alpha \in (0, 1)$ is the distribution parameter, and $\rho \leq 1$ represents the degree of substitutability between inputs.

4. **Returns to scale (advanced).** Consider generalized Cobb-Douglas production function
   $$F(K, L) = AK^\alpha L^\beta$$
   Under what conditions for α and β does it display constant returns to scale? What about increasing and decreasing returns to scale?

5. **Returns to scale with multiple factors.** We have so far focused on production function with two inputs. Now consider production function which also includes land:
   $$Y_t = K_t^\alpha L_t^\beta Land_t^{1-\alpha-\beta}$$
   Assuming that $\alpha + \beta < 1$, does this production function display constant returns to scale in capital and labour? Given that real world data do support constant returns to scale to inputs, what does this teach us about the importance land in production?

6. **Other properties of linear production function.** Consider the linear production function $Y = A \cdot (L + K)$. We have shown that it does satisfy the constant returns to scale properties. Does it satisfy the other properties discussed during lecture?

7. **Inada conditions (advanced).** Consider production function $Y = AK + BK^{\alpha}L^{1-\alpha}$, where $\alpha < 1$ and $A + B = 1$.

   (a) Show that this production function exhibits constant returns to scale.

   (b) Show that this production function exhibits diminishing marginal product of capital and labour.

   (c) Show that this production function satisfies both Inada conditions for labour.

   (d) Show that this production function satisfies one Inada condition for capital, but fails the other one.

   (e) What does this teach us about the typical aspect of production functions that fail this Inada condition? [Hint: Consider this in context of question 6.]

   (f) Draw the standard Solow growth graph with this production function.

8. **Essentiality.** It can be shown that the assumptions we made for our production function during lectures imply property called essentiality.[1] This can be mathematically stated as $F(K, 0) = 0$ and $F(0, L) = 0$.

   (a) Interpret what does this mean in words.

   (b) For both conditions provide (real or fictitious) counterexample showing it does not need to hold in reality.

9. **No labour.** Consider a closed economy with neoclassical production function. Is it possible in such economy that nobody would be working, for example because everybody inherited immense amount of wealth (e.g. large sums of money or property)? Explain why.

[1] If you want a challenge, you might want to try to prove it, possibly with ChatGPT, but its really not something useful.

Exercise 2: Basic Solow model

1. **Example with Cobb-Douglas production function.** Consider the basic Solow growth model with Cobb-Douglas production function $F(K, L) = K^{\alpha} L^{1-\alpha}$.

   (a) Derive per capita production function as well as the fundamental equation of Solow model.

   (b) Derive the expression for steady state in per capita terms, including the output, consumption, investment and depreciation.

   (c) Derive the expression for steady state in total terms.

   (d) Assume that there are two countries, one with $\delta = 0.2$ and $s = 0.1$ and one with $\delta = 0.2$ and $s = 0.3$. Assume that both have $\alpha = 1/3$. Calculate and compare their steady states levels of output, consumption, investment and depreciation per capita.

   (e) Suppose that both countries start off with a capital stock per worker of 1. What are the levels of income per worker and consumption per worker?

   (f) Follow from previous question. How much does the capital, output, consumption and investment per worker increase in the first year in both countries?

   (g) Now assume that there are three countries that have different value of α: one with $\alpha = 1/3$, one with $\alpha = 1/2$ and one with $\alpha = 2/3$. They all have the same savings and depreciation rate $\delta = 0.2$ and $s = 0.1$. What is the steady state value of output and consumption per capita in each country? What does that teach us about the role of α? Is it better to have higher or lower α? Explain.

2. **Energy as input.** Consider standard Solow growth model, but with an additional factor, energy. Let's assume that the production function has a Cobb-Douglas form, $F(K, L, E) = AK^{\alpha} L^{\beta} E^{1-\alpha-\beta}$. Answer following questions.

   (a) Does this function display constant returns to scale in capital and labour?

   (b) Does this function display constant returns to scale in capital, labour and energy?

   (c) Consider that we are currently producing a given amount of output, Y_0 using amounts of inputs K_0,L_0 and E_0. Now assume that due to green transition we need to use less energy, i.e. $E_1 < E_0$. What is going to be the effect on output?

   (d) Without deriving the steady state, what do you think will be the effect on steady state level of capital? Explain. [Hint: Think about the marginal product of capital before and after.]

3. **Human capital.** Consider an economy where output Y_t is produced using physical capital K_t, human capital H_t, and labor L_t. The production function is:

$$Y_t = K_t^{\alpha} H_t^{\beta} L_t^{1-\alpha-\beta}$$

   where $0 < \alpha < 1$ and $0 < \beta < 1$ are the output elasticities of physical capital and human capital, respectively. The two types of capital evolve as follows:

$$K_{t+1} = s_K Y_t + (1 - \delta) K_t$$
$$H_{t+1} = s_H Y_t + (1 - \delta) H_t$$

   (a) Derive the expression for per capita output as well as accumulation equations for per capita physical and human capital.

   (b) Derive the steady-state levels of physical capital k^*, human capital h^*, and output y^* as functions of the parameters.

   (c) Analyze how changes in the savings rates s_K and s_H affect the steady-state levels of k^*, h^*, and y^*.

4. **Population shock.** Consider economy described by basic Solow model. Consider that Thanos succeeds in obtaining the infinity stones and makes half of the people disappear.

   (a) Focus first on comparative statics. How will the new steady state compare with the old steady state in terms of capital/consumption/investment and output *per capita*?

   (b) Provide arguments why this prediction might prove too optimistic and why it might prove too pessimistic. How do these predictions relate to property of constant returns to scale?

   (c) How will the steady states compare in terms of absolute sizes (i.e. not per capita)?

   (d) Now focus on the dynamics. How will capital per capita evolve during transition from old to new steady state? What are the forces causing this?

   (e) Use previous answer to draw exactly the paths for output/consumption/investment during transition from old to new steady state.

5. **Climate change and depreciation rate.** Consider the issue of climate change and climate transition in context of basic Solow model and answer following questions.

   (a) What will be the effect of climate change leading to more extreme weather events like huricanes on depreciation rate?

   (b) What will be the effect of government-led climate transition on depreciation rate?

   (c) How will these effects affect the steady-state output and consumption? What about investment?

   (d) Now imagine that society decided to keep the same amount of capital per capita throughout the transition. What will this imply for investment and consumption?

6. **Effect of capital share.** One thing we have not discussed in lecture is the comparative statics of α. Use the standard Cobb-Douglas production function and show that capital per worker increases when α increases. What is the limiting implication of increasing α to 1? Provide intuition for these results. [Hint: Doing this via mathematical derivations is not easy, so it is better to approach this by investigating the formula for steady state capital per worker. Consult ChatGPT for the mathematics, if you are curious.]

7. **Numerical example of Solow model dynamics.** Continue from question 1, where we calculated the steady state value of capital/output/investment/consumption per worker under assumption of Cobb-Douglas production functions and values $\delta = 0.2$, $s = 0.1$ and $\alpha = 1/3$. Imagine that we start from the steady state and then the savings rate decreases from 0.1 to 0.05.

   (a) Does anything happen to capital per worker in the first period (i.e. in the period when savings rate changed *before we produce new capital*)?

   (b) Does the output per worker change in the period when savings rate changed? Link your answer to previous answer.

   (c) Calculate the investment per worker in the first period and compare it with the initial steady state value of investment. Explain why did it change even though the output did not change.

   (d) Calculate the investment per worker in the first period and compare it with the initial steady state value of investment. Explain why did it change even though the output did not change.

   (e) Calculate the depreciated capital per worker in the first period. How does it compare with the initial steady state amount of depreciated capital?

   (f) Why does investment per worker decline but depreciated capital per worker does not? Link your answer to answer to question (a).

(g) What is the amount of capital per worker in the begining of second period (i.e. after we produced new capital)? How does it compare to the initial steady state level of capital? Link your answer to answers to questions (c) and (e).

(h) What is the output per worker in the second period? Compare it with answer to question (b) and explain the differences.

(i) Calculate the investment per worker in the second period and compare it with investment in first period. Contrast why did it change this time around and why did it change in first period.

(j) Do the same for consumption per worker.

(k) Calculate the depreciated capital per worker in the second period. How does it compare with the amount of depreciated capital in first period?

(l) Why does the amount of depreciated capital per worker decline in second period and not in the first period?

(m) Is the difference between investment per worker and depreciated capital per worker larger in first or second period? Explain using the standard Solow model graph.

(n) What is the amount of capital per worker at the end of second period (i.e. after we produced new capital)?

(o) Did the amount of capital decline more in first period or in second period? Explain with reference to question (m).

(p) Based on your analysis here draw the whole path of capital/output/investment/depreciation/consumption per worker after the shock.

8. **Dynamics of Solow model.** Consider basic Solow model. For each of the following cases draw the full dynamic path of capital/output/investment/depreciation/consumption per worker like we did in the lecture.

   (a) Increase in savings rate when we were initially above the golden level of capital.

   (b) Decrease in savings rate when we were initially above the golden level of capital.

   (c) Jump in population.

   (d) Increase in depreciation rate like in Question 5.

   (e) Decrease in level of technology.

Exercise 3: Solow growth model with population

1. **Steady state.** Use the adjusted fundamental equation of Solow growth model with population and assume the Cobb-Douglas production function. Derive the expression for steady state output/consumption/investment and capital per worker.

2. **Demographic dividend.** There is a phenomenon called the demographic dividend, which is situation when economic growth increases because of shifts in a population's age structure, especially when the birth rate decreases from high levels. While Solow growth model cannot shed light on the role of age structure, it has something to say about the effect of moderation birth rate.

   (a) What is the effect of of moderation in birth rate on living standards? Explain.

   (b) What is the effect on overall growth rates of the economy? Explain the different from previous question.

3. **Dilution of capital.** Use the Solow model with population growth to answer following questions.

   (a) Provide a real world example how increasing population leads to higher need for investment to ensure sufficient capital per worker.

   (b) Provide a real world example how decreasing population could lead to higher consumption thanks to lower need for investment to replenish depreciating capital.

   (c) Now imagine that due to political reasons the country is unwilling to decrease the absolute amount of its capital (e.g. railroads, roads and post offices). What will this mean for consumption per capita? Compare your answer to previous situation as well as with situation when population is constant.

4. **Role of constant returns to scale.** In the basic Solow model, population growth leads to steady-state growth in total output, but not in output per worker. Do you think this would still be true if the production function exhibited increasing returns to scale? What about decreasing returns to scale? Explain you answers.

5. **Declining population.** Many developed countries face situation of declining population, with the most extreme example being South Korea, where on current trends will halve roughly every 50 years. There are political reasons why this is problematic, but let's focus only on economics.

   (a) What does this imply for the living standards according to the Solow growth model?

   (b) This feels strange as it is contrary to the general notion that declining population is an economic problem. One reason why the answer from Solow growth model is incomplete is because it does not consider the age distribution of population and what that implies (e.g. problem of too few young people to take care of old people). Can you think of other reason? [Hint: Think about what we have learned in question 4]

Exercise 4: Solow growth model with technological progress

1. **Example with Cobb-Douglas production function.** Suppose that the economy's production function is
$$Y = K^{\alpha}(EL)^{1-\alpha}$$
that the saving rate, s, is equal to 16%, and that the rate of depreciation, d, is equal to 10% and $alpha = 1/3$. Suppose further that the number of workers grows at 2% per year and that the rate of technological progress is 4% per year.
   (a) Derive the expression for steady state in per effective worker terms, including the output, consumption, investment and depreciation.
   (b) Derive the expression for output in per worker terms.
   (c) Derive the expression for total output.
   (d) Find the steady-state values for following variables.
      i. The capital stock per effective worker
      ii. Output per effective worker
      iii. The growth rate of output per effective worker
      iv. The growth rate of output per worker
      v. The growth rate of output
   (e) Suppose that the rate of technological progress halves to 2% per year. Compare the results with previous situation. Explain the differences.
   (f) Now suppose that the rate of technological progress is still equal to 4% per year, but the number of workers now grows at 0% per year. Compare the results with original situation. Explain the differences.
2. **Increase in savings rate.** Suppose the government enacts legislation that encourages saving and investment, such as the research tax credits. As a result, suppose the savings/investment rate jumps permanently from s_0 to s_1. Assume that the economy is initially on balanced growth path, with no population growth and growth of technology g, and assume that this growth does not change.
   (a) Derive how does the steady state capital per effective worker change as result of this change.
   (b) Sketch the dynamic path for capital per effective worker following this change. Do the same for investment and consumption per effective worker.
   (c) Now focus on per actual worker terms. Sketch the dynamic path for capital, output, investment and consumption per worker.
   (d) What is the takeaway from this exercise? Does the policy change permanently increase the *level* or the *growth rate* of output per worker?
   (e) How are things different from the basic Solow model without technological change?
3. **Increased pace of technological progress.** Consider the Solow growth model without population growth but with technological progress. Suppose that there thanks to invention of generative AI there is a permanent increase in the rate of technological progress, so that g rises from g_0 to g_1.
   (a) Derive how does the steady state capital per effective worker change as result of this change.
   (b) Sketch the dynamic path for capital per effective worker following this change. Do the same for investment and consumption per effective worker.
   (c) Now focus on per actual worker terms. Sketch the dynamic path for capital, output, investment and consumption per worker.

Exercise 5: Endogenous technological progress

1. **AK model.** Consider the AK model from lecture slides.
   (a) Show the properties of the production function.
   (b) Do the standard derivation of Solow growth model dynamics.
2. **Linear production function.** Consider linear production function $Y = A \cdot (L + K)$ and assume that population is constant. Answer following questions.
   (a) Show that this function does not display diminishing returns to capital. Explain intuitively why that is the case. Can you think of (real-world of fictitious) example when this production function might apply?
   (b) Assume that $A > \delta$. What is the steady-state level of per capita capital. Provide graphical analysis of your answer.
   (c) Now follow the standard derivation of Solow growth model dynamics and show that there is no steady state. [Hint: You can set $\Delta k = 0$ and derive the formula for steady state and show that it is nonsensical.]

Exercise 6: Growth accounting

1. **Cobb-Douglas example.** Consider the standard Cobb-Douglas production function and derive the growth accounting formula for the case of Hicks neutral technology and the case of Harrod neutral (labour-augmenting) technology. How are the two measures of technology related?

2. **Numerical example.** Consider the following data on the economy of Country X over a 10-year period:

Year	Output Growth (%)	Capital Growth (%)	Labor Growth (%)	Productivity Growth (%)
1	4.0	5.0	2.0	1.0
2	3.5	4.5	1.5	1.0
3	4.2	4.8	1.8	1.2
4	3.8	4.2	1.4	1.2
5	4.1	4.6	1.9	1.3
6	3.6	4.0	1.6	1.2
7	4.0	4.5	1.7	1.4
8	3.9	4.4	1.5	1.3
9	4.3	4.9	1.8	1.5
10	3.7	4.1	1.6	1.1

   (a) Using the Cobb-Douglas production function with a capital share of 0.3 and labor share of 0.7, calculate the contributions of capital, labor, and productivity (Total Factor Productivity, TFP) to output growth for each year.

   (b) Now imagine that you were not provided the last column. Using the Solow residual method, estimate the Total Factor Productivity (TFP) growth given the growth rates of output, capital, and labor, and the factor shares.

3. **Natural resources and measurement of technology.** Consider an economy where output Y_t is produced using physical capital K_t, labor L_t and natural resources N_t. The production function is:

$$Y_t = A_t K_t^{\alpha} L_t^{\beta} N_t^{1-\alpha-\beta}$$

   Consider two different countries, one with very low values of N_t and one with very high values of N_T. Answer following questions.

   (a) Assuming that levels of A_t and k_t are the same, which country will have higher output per capita?

   (b) Now imagine that it is not two different countries, but one country before and after discovery of large oil deposits, such as Guyana. Imagine you will perform growth accounting to estimate the level of technology, but you include only information on physical capital and labour. What will be the implication for your measure of A_t?

   (c) What does this imply about measures of technology. Consider two cases: (i) estimating technology at given time across countries, (ii) estimating contribution of technology to growth over time. In the second case be careful to consider how it depends on whether amount of natural resources changes or not.

   (d) Now drop the assumption that levels of A_t and k_t are the same. Is it true that countries with more natural resources will always have higher level output per capita? Explain.

   (e) In the world there are many countries with high levels of N_t but low levels of output per capita, so that the is negative relationship between natural resources and output per capita (at least in parts of the sample). What does this imply about relationship between A_t and N_t. Can you explain this?